Derived Algebraic Geometry

by

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Abstract

The purpose of this document is to establish the foundations for a theory of derived algebraic geometry based upon simplicial commutative rings. We define derived versions of schemes, algebraic spaces, and algebraic stacks. Our main result is a derived analogue of Artin's representability theorem, which provides a precise criteria for the representability of a moduli functor by geometric objects of these types.

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Contents

Chapter 1

Introduction

1.1 Bezout's Theorem

Let $C, C' \subseteq \mathbf{P}^2$ be two smooth algebraic curves of degrees n and m in the complex projective plane $\mathbf{P}^2$. If C and C' meet transversely, then the classical theorem of Bezout (see for example [10]) asserts that $C \cap C'$ has precisely nm points.

We may reformulate the above statement using the language of cohomology. The curves C and C' have fundamental classes $[C], [C'] \in \mathrm{H}^2(\mathbf{P}^2, \mathbf{Z})$. If C and C' meet transversely, then we have the formula

$$[C] \cup [C'] = [C \cap C'],$$

where the fundamental class $[C \cap C'] \in \mathrm{H}^4(\mathbf{P}^2, \mathbf{Z}) \simeq \mathbf{Z}$ of the intersection $C \cap C'$ simply counts the number of points in the intersection. Of course, this should not be surprising: the cup-product on cohomology classes is defined so as to encode the operation of intersection. However, it would be a mistake to regard the equation $[C] \cup [C'] = [C \cap C']$ as obvious, because it is not always true. For example, if the curves C and C' meet nontransversely (but still in a finite number of points), then we always have a strict inequality

$$[C] \cup [C'] > [C \cap C']$$

if the right hand side is again interpreted as counting the number of points in the set-theoretic intersection of C and C'.

If we want a formula which is valid for non-transverse intersections, then we must alter the definition of $[C \cap C']$ so that it reflects the appropriate intersection multiplicities. Determination of these intersection multiplicities requires knowledge of the intersection $C \cap C'$ as a scheme, rather than simply as a set. This is one of the classical arguments that nonreduced scheme structures carry useful information: the intersection number $[C] \cup [C'] \in \mathbf{Z}$, which is defined *a priori* by perturbing the curves so that they meet transversally, can also be computed directly (without perturbation) if one is willing to contemplate a potentially non-reduced scheme structure on the intersection.

In more complicated situations, the appropriate intersection multiplicities cannot always be determined from the scheme-theoretic intersection alone. Suppose that C and C' are (possibly singular) subvarieties of $\mathbf{P}^n$, of complementary dimension and having a zero-dimensional intersection. In this case, the appropriate intersection number associated to a point $p \in C \cap C'$ is not always given by the complex dimension of the local ring

$$\mathcal{O}_{C \cap C', p} = \mathcal{O}_{C,p} \otimes_{\mathcal{O}_{\mathbf{P}^n,p}} \mathcal{O}_{C',p}.$$

The reason for this is easy to understand from the point of view of homological algebra. Since the tensor product functor $\otimes_{\mathcal{O}_{\mathbf{P}^n,p}}$ is not exact, it does not have good properties when considered alone. According to Serre's intersection formula, the correct intersection multiplicity is instead the Euler characteristic

$$\sum (-1)^i \dim \mathrm{Tor}_i^{\mathcal{O}_{\mathbf{P}^n,p}}(\mathcal{O}_{C,p}, \mathcal{O}_{C',p}).$$

This Euler characteristic contains the dimension of the local ring of the scheme-theoretic intersection as its leading term, but also higher-order corrections. We refer the reader to [31] for further discussion of this formula for the intersection multiplicity.

If we would like the equation $[C] \cup [C'] = [C \cap C']$ to remain valid in the more complicated situations described above, then we will need to interpret the intersection $C \cap C'$ in some way which remembers not only the tensor product $\mathcal{O}_{C,p} \otimes_{\mathcal{O}_{\mathbf{P}^n,p}} \mathcal{O}_{C',p}$, but the higher Tor terms as well. Moreover, we should not interpret these Tor-groups separately, but rather should think of the total derived functor $\mathcal{O}_{C,p} \otimes^L_{\mathcal{O}_{\mathbf{P}^n,p}} \mathcal{O}_{C',p}$ as a kind of "generalized ring".

These considerations lead us naturally to the subject of *derived algebraic geometry*. Using an appropriate notion of "generalized ring", we will mimic the constructions of classical scheme theory to obtain a theory of *derived schemes* in which a version of the formula $[C] \cup [C'] = [C \cap C']$ can be shown to hold with (essentially) *no* hypotheses on C and C'. Here, we must interpret the intersection $C \cap C'$ in the sense of derived schemes, and we must take great care to give the proper definition for the fundamental classes (the so-called *virtual fundamental classes* of [4]).

To motivate our discussion of "generalized rings", we begin by considering the simplest case of Bezout's theorem, in which C and C' are lines in the projective plane $\mathbf{P}^2$. In this case, we know that $[C] \cup [C']$ is the cohomology class of a point, and that C intersects C' transversely in one point so long as C and C' are distinct. However, when the equality $C = C'$ holds, the scheme-theoretic intersection $C \cap C'$ does not even have the correct dimension.

Let us now try to give an idea of how we might formulate a definition for "derived scheme-theoretic intersections" which will handle the degenerate situation in which $C = C'$. For simplicity, let us consider only lines in the affine plane $\mathbf{A}^2 \subseteq \mathbf{P}^2$, with coordinate ring $\mathbf{C}[x, y]$. Two distinct lines in $\mathbf{A}^2$ may be given by equations $x = 0$ and $y = 0$. The scheme-theoretic intersection of these two lines is the spectrum of the ring $\mathbf{C}[x, y]/(x, y) \simeq \mathbf{C}$, obtained from $\mathbf{C}[x, y]$ by setting the equations of both lines equal to zero. It has dimension zero because $\mathbf{C}[x, y]$ is two-dimensional to begin with, and we have imposed a total of two equations.

Now suppose that instead of C and C' being two distinct lines, they are actually two *identical* lines, both of which have the equation $x = 0$. In this case, the affine ring of the scheme theoretic intersection is given by $\mathbf{C}[x,y]/(x,x) \simeq \mathbf{C}[y]$. This ring has dimension one, rather than the expected dimension zero, because the two equations are not independent: setting $x=0$ twice is equivalent to setting $x=0$ once. To obtain derived algebraic geometry, we need a formalism of "generalized rings" in which imposing the equation $x=0$ twice is not equivalent to imposing the equation once.

One way to obtain such a formalism is by "categorifying" the notion of a commutative ring. That is, in place of ordinary commutative rings, we should consider *categories* equipped with "addition" and "multiplication" operations (which are now functors, rather than ordinary functions). For purposes of the present discussion, let us call such an object a *categorical ring*. We shall not give a precise axiomatization of this notion, which turns out to be quite complicated (see [19], for example).

Example 1.1.1. Let $\mathbf{Z}_{\geq 0}$ denote the semiring of nonnegative integers. We note that $\mathbf{Z}_{\geq 0}$ arises in nature through the process of "decategorification". The nonnegative integers were introduced in order to count finite collections: in other words, they correspond to isomorphism classes of objects in the category $\mathcal{Z}$ of finite sets. Then $\mathcal{Z}$ is naturally equipped with the structure of a *categorical semiring*, where the addition is given by forming disjoint unions and the multiplication is given by Cartesian products. (In order to complete the analogy with the above discussion, we should "complete" the category $\mathcal{Z}$ by formally adjoining inverses, to obtain a categorical ring rather than a categorical semiring, but we shall ignore this point.)

To simplify the discussion, we will consider only categorical rings which are groupoids: that is, every morphism in the underlying category is an isomorphism. If $\mathcal{C}$ is a categorical ring, then the collection of isomorphism classes of objects $\pi_0\,\mathcal{C}$ of $\mathcal{C}$ forms an ordinary ring. Every commutative ring R arises in this way: for example, we may take $\mathcal{C}_R$ to be a category whose objects are the elements of R and which contains only identity maps for morphisms. The categorical rings which arise in this way are very special: their objects have no nontrivial automorphisms. For a given commutative ring R, there are many other ways to realize an isomorphism $R \simeq \pi_0\,\mathcal{C}$ with the collection of isomorphism classes of objects in a categorical ring $\mathcal{C}$. A crucial observation to make is that although $\mathcal{C}$ is not uniquely determined by R, there is often a natural choice for $\mathcal{C}$ which is dictated by the manner in which R is constructed.

As an example, let us suppose that the commutative ring R is given as a quotient $R'/(x-y)$, where R' is some other commutative ring and $x, y \in R'$ are two elements. Suppose that the ring R' has already been "categorified" in the sense that we have selected some categorical ring $\mathcal{C}'$ and an identification of R' with $\pi_0\,\mathcal{C}'$. To this data, we wish to associate some "categorification" $\mathcal{C}$ of R. Roughly, the idea should be to think of x and y objects of $\mathcal{C}'$, and to impose the relation $x=y$ at the *categorical* level. However, it is extremely unnatural to ask that two objects in a category be *equal*; instead one should ask that they be *isomorphic*. In other words, the quotient category $\mathcal{C}$ should not be obtained from $\mathcal{C}'$ by identifying the objects x and y. Instead we should construct $\mathcal{C}$ by *enlarging* $\mathcal{C}'$ so that it includes an isomorphism $\alpha : x \simeq y$. Since we want $\mathcal{C}$ to be a categorical ring, the formation of this enlargement is a somewhat complicated business: in addition to the new isomorphism α, we must also adjoin other isomorphisms which can be obtained from α through addition, multiplication, and composition (and new relations, which may cause distinct isomorphisms in $\mathcal{C}'$ to have the same image in $\mathcal{C}$).

To make the connection with our previous discussion, let us note that the construction of $\mathcal{C}$ from $\mathcal{C}'$ given in the preceding paragraph is interesting even in the "trivial" case where $x = y$. In this case, x and y are already isomorphic when thought of as objects of $\mathcal{C}'$. However, in $\mathcal{C}$ we get a *new* isomorphism α between x and y, which generally does not lie in the image of the natural map $\mathrm{Hom}_{\mathcal{C}'}(x, y) \rightarrow \mathrm{Hom}_{\mathcal{C}}(x, y)$. Consequently, even though the natural quotient map $R' \rightarrow R$ is an isomorphism, the corresponding "categorical ring homomorphism" $\mathcal{C}' \rightarrow \mathcal{C}$ need not be an equivalence of categories. Imposing the new relation $x = y$ does not change the collection of isomorphism classes of objects, but usually *does* change the automorphism groups of the objects. Consequently, if we begin with *any* objects x and y, we can iterate the above construction two or more times, to obtain a categorical ring $\mathcal{C}$ equipped with multiple isomorphisms $x \simeq y$. These isomorphisms are (in general) distinct from one another, so that the categorical ring $\mathcal{C}$ "knows" how many times x and y have been identified.

We have now succeeded in finding a formalism which is sensitive to "redundant" information: we just need to replace ordinary commutative rings with categorical rings. The next question we should ask is whether or not this formalism is of any use. Let us suppose that, in the above situation, $\mathcal{C}'$ is discrete in the sense that every object has a trivial automorphism group. We note that the ring $R = R'/(x - y)$ of objects of $\mathcal{C}$ may be naturally identified with the cokernel of the map

$$\phi : R' \stackrel{x-y}{\rightarrow} R'.$$

It turns out that the automorphism groups in $\mathcal{C}$ also carry interesting information: they all turn out to be naturally isomorphic to the kernel of ϕ.

Let us return to geometry for a moment, and suppose that R' is the affine ring of a curve (possibly nonreduced) in $\mathbf{A}^2 = \operatorname{Spec} \mathbf{C}[x, y]$. Let $R'' = \mathbf{C}[x, y]/(x - y)$ denote the affine ring of the diagonal. Then the cokernel and kernel of ϕ may be naturally identified with $\mathrm{Tor}_0^{\mathbf{C}[x,y]}(R', R'')$ and $\mathrm{Tor}_1^{\mathbf{C}[x,y]}(R', R'')$. In other words, just as the leading term in Serre's intersection formula has a geometric interpretation in terms of tensor constructions with ordinary commutative rings, we can obtain a geometric interpretation for the second term if we are willing to work with categorical rings.

Unfortunately, this is far as categorical rings will take us. In order to interpret the next term in Serre's intersection formula, we would need to take "categorification" one step further and consider ring structures on *2-categories*. If we want to understand the entire formula, then we need to work with ∞-categories. Fortunately, the ∞-categorical rings which we will need are of a particularly simple flavor: they are *∞-groupoids*, meaning that all of the n-morphisms are invertible for $n \geq 1$. Although the general theory of ∞-categories is a hairy business, the ∞-groupoids are well-understood: they are essentially the same thing as *spaces* (say, CW-complexes), as studied in homotopy theory. If X is any space, then it gives rise to an ∞-groupoid as follows: the objects are the points of X, the morphisms are the paths between points, the 2-morphisms are homotopies between paths, the 3-morphisms are homotopies between homotopies, and so on. The converse assertion, that every ∞-groupoid arises in this way, is a generally accepted principle of higher category theory.

This suggests that an ∞-categorical ring should be a topological space X equipped with some kind of ring structure. The simplest way of formulating the latter condition is to require X to be a topological ring: that is, a commutative ring with a topology, for which the addition and multiplication are continuous maps.

Remark 1.1.2. There exist other reasonable theories of "∞-categorical rings", in which the ring axioms need only hold only up to homotopy. In fact, the setting of topological commutative rings turns out to rather restrictive: the categorical semiring $\mathcal{Z}$ of finite sets, discussed above, cannot be modelled by a topological semiring. This is true even after passing to a categorical ring by formally adjoining "negatives". We will survey the situation in §2.6, where we argue that topological commutative rings seem better suited to algebro-geometric purposes than their more sophisticated relatives.

Just as an ordinary scheme is defined to be "something which looks locally like $\operatorname{Spec} A$ where A is a commutative ring", a derived scheme will be defined to be "something which looks locally like $\operatorname{Spec} A$ where A is a topological commutative ring".

Remark 1.1.3. We should emphasize that the topology of such a ring A only matters "up to homotopy equivalence": it is simply a mechanism which allows us to discuss paths, homotopies between paths, and so forth. The topology on A should be thought of as an essentially combinatorial, rather than geometric, piece of data. Consequently, most of the topological rings which arise in mathematics are quite uninteresting from our point of view. For example, any ring which is a topological vector space over $\mathbf{R}$ is contractible, and thus equivalent to the zero ring. On the other hand, any $\mathfrak{p}$-adically topologized ring has no nontrivial paths, and is thus equivalent to a discrete ring from our point of view. The topological rings which *do* arise in derived algebraic geometry are generally obtained from discrete rings by applying various categorical constructions, and are difficult to describe directly.

The theory of derived algebraic geometry bears some similarity to the theory of algebraic stacks. Both theories involve some mixture of classical algebro-geometric ingredients (commutative algebra, sheaf theory, and so forth) with some additional ideas which are category-theoretic, or homotopy-theoretic, in nature. However, we should emphasize that the aims of the two theories are completely distinct. The main purpose for the theory of algebraic stacks is to provide a setting in which various moduli functors are representable (thereby enabling one to discuss, for example, a moduli stack of smooth curves of some fixed genus). This is *not* the case for derived algebraic geometry. Rather, one should think of the relationship between derived schemes and ordinary schemes as analogous to the relationship between ordinary schemes and reduced schemes. If one considers only reduced test objects, then non-reduced schemes structures are of no help in representing moduli functors because

$\mathrm{Hom}(X, Y^{\mathrm{red}}) \xrightarrow{\sim} \mathrm{Hom}(X, Y)$ whenever X is reduced. The theory of non-reduced schemes is instead useful because it enlarges the class of test objects on which the moduli functors are defined. Even if our ultimate interest is only in reduced schemes (such as smooth algebraic varieties), it is useful to consider these schemes as defining functors on possibly non-reduced rings. For example, the non-reduced scheme $X = \mathrm{Spec}\, \mathbf{C}[\epsilon]/(\epsilon^2)$ is an interesting test object which tells us about tangent spaces: $\mathrm{Hom}(X, Y)$ may be thought of as classifying tangent vectors in Y.

The situation for derived schemes is similar: assuming that our moduli functors are defined on an even larger class of test objects leads to an even better understanding of the underlying geometry. We will illustrate this using the following example from deformation theory:

Example 1.1.4. Let X be a smooth projective variety over the complex numbers. The following statements about the deformation theory of X are well-known:

1. The first-order deformations of X are classified by the cohomology $\mathrm{H}^1(X, T_X)$ of X with coefficients in the tangent bundle of X.

2. A first-order deformation of X extends to a second-order deformation if and only if a certain obstruction in $\mathrm{H}^2(X, T_X)$ vanishes.

Assertion (1) is very satisfying: it provides a concrete geometric interpretation of an otherwise abstract cohomology group, and it can be given a conceptual proof using the interpretation of H^1 as classifying torsors. In contrast, (2) is usually proven by an ad-hoc argument which uses the local triviality of the first order deformation to extend locally, and then realizes the obstruction as a cocycle representing the (possible) inability to globalize this extension. This argument is computational rather than conceptual, and it does give not us a geometric interpretation of the cohomology group $\mathrm{H}^2(X, T_X)$. We now sketch an alternative argument for (2) which does not share these defects.

As it turns out, $\mathrm{H}^2(X, T_X)$ also classifies a certain kind of deformation of X, but a deformation of X over the "nonclassical" base $\mathrm{Spec}\, \mathbf{C}[\delta]$ where we adjoin a generator δ in "degree 1" (in other words, we take the ordinary ring $\mathbf{C}$ and impose the equation $0 = 0$ according to the recipe outlined earlier). Namely, elements of $\mathrm{H}^2(X, T_X)$ may be identified with equivalences classes of flat families over $\mathrm{Spec}\, \mathbf{C}[\delta]$ together with an identification of the closed fiber of the family with X. In other words, $\mathrm{H}^2(X, T_X)$ classifies $\mathrm{Spec}\, \mathbf{C}[\delta]$-valued points of some moduli stack of deformations of X.

The interpretation of obstructions as elements of $\mathrm{H}^2(X, T_X)$ can be obtained as follows. The ordinary ring $\mathbf{C}[\epsilon]/(\epsilon^3)$ can be realized as a "homotopy fiber product" $\mathbf{C}[\epsilon]/(\epsilon)^2 \times_{\mathbf{C}[\delta]} \mathbf{C}$, for an appropriately chosen map of "generalized rings" $\mathbf{C}[\epsilon]/(\epsilon^2) \to \mathbf{C}[\delta]$. In geometric terms, this means that $\mathrm{Spec}\, \mathbf{C}[\epsilon]/(\epsilon^3)$ may be constructed as a pushout $\mathrm{Spec}\, \mathbf{C}[\epsilon]/(\epsilon^2) \coprod_{\mathrm{Spec}\, \mathbf{C}[\delta]} \mathrm{Spec}\, \mathbf{C}$. Therefore, to give a second-order deformation of X, we must give X, a first order deformation of X, and an identification of their restrictions to $\mathrm{Spec}\, \mathbf{C}[\delta]$. This is possible if and only if the first order deformation of X restricts to the trivial deformation of X over $\mathrm{Spec}\, \mathbf{C}[\delta]$, which is equivalent to the vanishing of a certain element of $\mathrm{H}^2(X, T_X)$.

Derived algebraic geometry seems to be the appropriate setting in which to understand the deformation-theoretic aspects of moduli problems. It has other applications as well, many of which stem from the so-called "hidden smoothness" philosophy of Kontsevich. According to this point of view, if one works entirely in the context of derived algebraic geometry, one can (to some extent) pretend that all algebraic varieties are smooth. More precisely, many constructions which are usually discussed only in the smooth case can be adapted to nonsmooth varieties using ideas from derived algebraic geometry:

- The cotangent bundle of a smooth algebraic variety may be generalized to the non-smooth case as the *cotangent complex.*

- The deRham complex of a smooth algebraic variety can be generalized to the non-smooth case as the *derived deRham complex* of Illusie (see [17]).

- In certain cases, one can mimic the usual construction of the sheaf of differential operators on a smooth variety, using the tangent complex in place of the tangent bundle. This allows one to formulate a theory of (complexes of) algebraic $\mathcal{D}$-modules on a possibly singular algebraic variety X, whose definition does not depend on (locally) embedding X into a smooth ambient variety.

- The fundamental class of an algebraic variety may be replaced by a more subtle "virtual fundamental class", which allows one to prove a Bezout-type theorem $[C] \cup [C'] = [C \cap C']$ in complete generality.

Remark 1.1.5. The freedom to compute with non-transverse intersections can be extremely useful, because interesting situations often possess symmetries which are lost after perturbation. As an example, consider equivariant cobordism theory. Because transversality fails in the equivariant context, the classical Pontryagin-Thom construction does not work as expected to produce equivariant spectra whose homotopy groups are cobordism classes of manifolds equipped with smooth group actions (see [14]). Consequently, one obtains two *different* notions of equivariant cobordism groups: one given by manifolds modulo cobordism, and one given by the Pontryagin-Thom construction. The second of these constructions seems to fit more naturally into the context of equivariant stable homotopy theory. The geometric meaning of the latter construction can be understood in the setting of *derived differential topology*: the Pontryagin-Thom construction produces a spectrum whose homotopy groups represent certain cobordism classes of equivariant *derived manifolds* (a class of objects which includes non-transverse intersections of ordinary manifolds). In the non-equivariant case, any derived manifold is cobordant to an ordinary manifold, but in the presence of a group action this is not true.

We hope that the reader is now convinced that a good theory of derived algebraic geometry would be a useful thing to have. The purpose of this paper is to provide the foundations for such a theory. We will discuss derived schemes (and, more generally, derived versions of Artin stacks) from both a geometric and functorial point of view. Our main result is an analogue of Artin's representability theorem, which gives a precise characterization of those functors which are representable by derived stacks. In [23] and [24] we shall forge the link between the formalism developed here and some of the applications mentioned above.

There exist other approaches to derived algebraic geometry in the literature. The earliest of these is the notion of a *differential graded scheme* (see [7], for example). This approach employs differential graded algebras in place of topological rings. In characteristic zero, the resulting theory can be related to ours. In positive characteristic, the notion of a differential graded scheme is poorly behaved. More recent work of Toën and Vezzosi has been based on the more sophisticated notion of an E_∞-ring spectrum. We will survey the relationship between these approaches in §2.6. It is worth noting that the proof of our main result, Theorem 7.1.6, can be adapted to produce moduli spaces in the E_∞-context.

1.2 Contents

We now outline the contents of this paper. After this introduction, we will begin in §2 by reviewing some of the background material that we shall need from the theory of abstract stable homotopy categories and structured ring spectra. Since these topics are somewhat technical and are adequately treated in the literature, our exposition has the character of a summary.

In §3, we begin to study the "generalized rings" of the introduction in their incarnation as *simplicial commutative rings*. We explain how to generalize many ideas from commutative algebra to this generalized setting, and review the theory of cotangent complexes. Finally, we discuss an analogue of Popescu's theorem on the smoothing of ring homomorphisms, which applies in the derived setting.

Our study of commutative algebra takes on a more geometric flavor in §4, where we discuss various topologies on simplicial commutative rings and the corresponding "spectrification" constructions. This leads us to the definition of a derived scheme, which we shall proceed to relate to the classical theory of schemes, algebraic spaces, and Deligne-Mumford stacks.

The geometric approach to scheme theory gives way in §5 to a more categorical approach. We show that derived schemes may also be described as certain space-valued functors defined on simplicial commutative rings. We then consider a more general class of functors, analogous to Artin stacks (and more generally, higher Artin stacks) in the classical setting. We follow this with a discussion of various properties of derived schemes, derived Artin stacks, and morphisms between them.

In §6, we will discuss the derived version of completions of Noetherian rings, and give a characterization of those functors which are representable by complete local Noetherian rings. This result is closely related to the infinitesimal deformation theory discussed in [30].

In §7, we give the proof of our main result, a derived version of Artin's representability theorem. We give a somewhat imprecise formulation as Theorem 1.2.1 below; the exact statement requires concepts which are introduced later and will be given as Theorem 7.5.1. The theorem addresses the question of when an abstract moduli functor $\mathcal{F}$ is representable by a geometric object, so that $\mathcal{F}(A) = \mathrm{Hom}(\mathrm{Spec}\, A, X)$ for some derived scheme or derived stack X. We note that even if $\mathcal{F}$ is represented by an ordinary scheme, it will not be a *set-valued* functor when we apply it to topological commutative rings. Hence, we consider instead *space-valued* functors.

Theorem 1.2.1. *Let R be a Noetherian ring which is excellent and possesses a dualizing complex (more generally, R could be a topological ring satisfying appropriate analogues of these conditions). Let $\mathcal{F}$ be a covariant functor from topological R-algebras to spaces (always assumed to carry weak homotopy equivalences into weak homotopy equivalences). We shall suppose that there exists an integer n such that $\pi_i(\mathcal{F}(A), p) = 0$ for any $i > n$, any* discrete *R-algebra A, and any base point $p \in \mathcal{F}(A)$ (if $n = 0$, this says that when A is discrete, $\mathcal{F}(A)$ is homotopy equivalent to a discrete space: in other words, $\mathcal{F}$ is* set-valued *when restricted to ordinary commutative rings).*

The functor $\mathcal{F}$ is representable by a derived stack which is almost of finite presentation over R if and only if the following conditions are satisfied:

1. *The functor $\mathcal{F}$ satisfies the functorial criterion for being almost of finite presentation (that is, it commutes with certain filtered colimits, up to homotopy).*
2. *The functor $\mathcal{F}$ is a sheaf with respect to the étale topology.*
3. *If $A \to C$ and $B \to C$ are fibrations of topological R-algebras which induce surjections $\pi_0 A \to \pi_0 C$, $\pi_0 B \to \pi_0 C$, then $\mathcal{F}(A \times_C B)$ is equivalent to the homotopy fiber product of $\mathcal{F}(A)$ and $\mathcal{F}(B)$ over $\mathcal{F}(C)$.*
4. *The functor $\mathcal{F}$ is nilcomplete (see §3.4); this is a harmless condition which is essentially always satisfied).*
5. *If A is a (discrete) commutative ring which is complete, local, and Noetherian, then $\mathcal{F}(A)$ is equivalent to the homotopy inverse limit of the sequence of spaces $\{\mathcal{F}(A/\mathfrak{m}^k)\}$, where $\mathfrak{m}$ denotes the maximal ideal of A.*
6. *Let $\eta \in \mathcal{F}(C)$, where C is a (discrete) integral domain which is finitely generated as a $\pi_0 R$-algebra. For each $i \in \mathbf{Z}$, the tangent module $T_i(\eta)$ (defined in §7.4) is finitely generated as a C-module.*

Our proof of this result follows Artin (see [2]), making use of simplifications introduced by Conrad and de Jong (see [8]) and further simplifications which become possible only in the derived setting.

We remark that the representability theorem is actually quite usable in practice. Of the six hypotheses listed above, the first four are usually automatically satisfied. Condition (5) stated entirely in terms of the restriction of the functor $\mathcal{F}$ to "classical" rings; in particular, if this restriction is representable by a scheme, algebraic space, or algebraic stack, then condition (5) is satisfied. Condition (6) is equivalent to the existence of a reasonable cotangent complex for the functor $\mathcal{F}$, which is a sort of linearized version of the problem of constructing

$\mathcal{F}$ itself. This linearized problem is usually easy to solve using the tools provided by derived algebraic geometry.

We conclude in §8 with some applications of our version of Artin's theorem. In particular, we define derived versions of Hilbert functor, the Picard functor, and the "stable curve" functor. Using our representability theorem, we will prove the representability of these functors and thereby construct derived analogues of Hilbert schemes, Picard schemes and moduli stacks of stable curves (some of these have been constructed in characteristic zero by very different methods; see [7]).

Throughout this paper, we will prove "derived versions" of classical results in commutative algebra and algebraic geometry, such as Popescu's theorem on smoothing ring homomorphisms, Grothendieck's formal GAGA theorem, and Schlessinger's criterion for the formal representability of "infinitesimal" moduli problems. These results are needed for our representability theorem and its applications, but only in their classical incarnations. Consequently, some of our discussion is unnecessary: in particular §6 might be omitted entirely. Our justification for including these results is that we feel that derived algebraic geometry can contribute to our understanding of them, either by offering more natural formulations of the statements (as in the case of Schlessinger's criterion) or more natural proofs (as in the case of the formal GAGA theorem).

1.3 Notation and Terminology

It goes without saying that the study of derived algebraic geometry requires a great deal of higher category theory. This is a story in itself, which we cannot adequately treat here. For a review of ∞-category theory from our point of view, we refer the reader to [22]. We will generally follow the terminology and notational conventions of [22] regarding ∞-categories. In particular, we shall write $\mathcal{S}$ for the ∞-category of spaces.

However, there is one bit of terminology on which we will not follow [22], and that is our use of the word "stack". The word "stack" has come to have several closely related meanings in mathematics: a "sheaf" of categories, a "sheaf" of groupoids, a geometric object which represents a groupoid-valued functor, and (in [22]) a "sheaf" of ∞-groupoids. In this paper, we shall use the word "stack" in the third sense: in reference to algebro-geometric objects. For all other purposes, we shall use the word "sheaf", together some indication of what sort of values are taken by the sheaf in question. *If not otherwise specified, all sheaves are assumed to be valued in the ∞-category $\mathcal{S}$ of spaces*, rather than in the ordinary category of sets.

We will also make occasional use of the theory of ∞-topoi developed in [22]. This is not entirely necessary: using Theorem 4.5.10, one can reformulate our notion of a derived scheme in a fashion which mentions only ordinary topoi. However, in this case we would still need to deal with $\mathcal{S}$-valued sheaves on topoi, and the language of ∞-topoi seems best suited to this purpose (see Remark 4.1.2).

If $\mathcal{C}$ is an ∞-category and $X \in \mathcal{C}$ is an object, then we will write $\mathcal{C}_{/X}$ for the slice ∞-category whose objects are diagrams $A \to X$ in $\mathcal{C}$. Dually, we write $\mathcal{C}_{X/}$ for the ∞-category whose objects are diagrams $X \to A$ in $\mathcal{C}$. Finally, given a morphism $f : X \to Y$ in $\mathcal{C}$, we write $\mathcal{C}_{X//Y}$ for the ∞-category $(\mathcal{C}_{X/})_{/Y} \simeq (\mathcal{C}_{/Y})_{X/}$.

We remark that for us, the ∞-category of $\mathcal{S}$-valued sheaves on a topos $\mathfrak{X}$ is *not* necessarily the one given by the Jardine model structure on simplicial presheaves. We briefly review the situation, which is studied at greater length in [22]. If $\mathcal{X}$ is an ∞-topos, then the full subcategory $\tau_{\leq 0}\,\mathcal{X} \subseteq \mathcal{X}$ consisting of discrete objects forms an ordinary (Grothendieck) topos. There is an adjoint construction which produces an ∞-topos $\Delta\mathfrak{Y}$ from any ordinary topos $\mathfrak{Y}$. The adjunction takes the form of a natural equivalence

$$\mathrm{Hom}(\mathcal{X}, \Delta\mathfrak{Y}) \simeq \mathrm{Hom}(\tau_{\leq 0}\,\mathcal{X}, \mathfrak{Y})$$

between the ∞-category of geometric morphisms (of ∞-topoi) from $\mathcal{X}$ to $\Delta\mathfrak{Y}$ and the category of geometric morphisms (of ordinary topoi) from $\tau_{\leq 0}\,\mathcal{X}$ to $\mathfrak{Y}$. The Jardine model structure on simplicial presheaves produces not the ∞-topos $\Delta\mathfrak{Y}$ but instead a localization thereof, which inverts the class of ∞-connected morphisms. Although this localization leads to simplifications in a few places, we feel that it is on the whole more natural to work with $\Delta\mathfrak{Y}$. In practice, the distinction will never be important.

Throughout this paper, we will encounter ∞-categories equipped with a tensor product operation $\otimes$. Usually this is related to, but not exactly a generalization of, some "ordinary" tensor product for modules over a ring. For example, if R is a commutative ring, then the left derived functors of the ordinary tensor product give rise to a tensor product operation $\otimes^L$ on the derived category of R-modules (and also on the ∞-category which gives rise to it). To avoid burdening the notation, we will omit the superscript. Thus, if M and N are R-modules, $M \otimes N$ will not denote the ordinary tensor product of M and N but instead the complex $M \otimes^L N$ whose homologies are the R-modules $\mathrm{Tor}_i^R(M, N)$. Whenever we need to discuss the ordinary tensor product operation, we shall denote it by $\mathrm{Tor}_0^R(M, N)$. We will use a similar notation for dealing with inverse limits of abelian groups. If $\{A_n\}$ is an inverse system of abelian groups, then it may be regarded as an inverse system of spectra, and it has a homotopy inverse limit which is a spectrum that shall be denoted by $\lim\{A_n\}$. The homotopy groups of this spectrum are given by the right derived functors of the inverse limit, and we shall denote them by $\lim^k\{A_n\} = \pi_{-k}\lim\{A_n\}$. We remark that if $\{A_n\}$ is given by a *tower*

$$\ldots \to A_2 \to A_1 \to A_0$$

of abelian groups, then $\lim^k\{A_n\}$ vanishes for $k \notin \{0, 1\}$.

We use the word *connective* to mean (-1)-connected; that is, a spectrum X is connective if $\pi_i X = 0$ for $i < 0$. We call a space or spectrum X *n-truncated* if $\pi_i X$ is trivial for $i > n$ (and any choice of base point). We call a space or spectrum *truncated* if it is k-truncated for some $k \in \mathbf{Z}$ (and therefore for all sufficiently large values of k).

Chapter 2

Background

The purpose of this section is to provide a brief introduction to certain ideas which will appear repeatedly throughout this paper, such as stable ∞-categories and structured ring spectra. Most of this material is adequately treated in the literature, so we generally be content to sketch the ideas without going into extensive detail.

2.1 Stable ∞-Categories

It has long been understood that there is a formal analogy between chain complexes with values in an abelian category and topological spaces (so that one speaks of *homotopies* between complexes, *contractible* complexes, and so forth). The analogue of the homotopy category of topological spaces is the derived category of an abelian category, a triangulated category which provides a good setting for many constructions in homological algebra. For some sophisticated applications, the derived category is too crude: it identifies homotopic morphisms of chain complexes without remembering *why* they are homotopic. In order to correct this defect, it is necessary to view the derived category as the homotopy category of some underlying ∞-category. We review how to do this in §2.3. It turns out that the ∞-categories which arise in this way have special properties which are related to the additive structure of the underlying triangulated category. The purpose of this section is to investigate ∞-categories with these special properties, which we shall call *stable* ∞-categories.

The notion of a stable ∞-category has been investigated in the context of model categories under the name of a *stable model category* (for a discussion, see [15]), and later in the more natural context of Segal categories.

Definition 2.1.1. Let $\mathcal{C}$ be an ∞-category. An object of $\mathcal{C}$ is a *zero object* if it both initial and final.

In other words, an object $0 \in \mathcal{C}$ is zero if $\mathrm{Hom}_{\mathcal{C}}(X, 0)$ and $\mathrm{Hom}_{\mathcal{C}}(0, X)$ are both contractible for any object $X \in \mathcal{C}$.

Remark 2.1.2. If $\mathcal{C}$ has a zero object, then that object is determined up to (essentially unique) equivalence.

Remark 2.1.3. Let $\mathcal{C}$ be an ∞-category with a zero object 0. For any $X, Y \in \mathcal{C}$, the natural map

$$\mathrm{Hom}_{\mathcal{C}}(X, 0) \times \mathrm{Hom}_{\mathcal{C}}(0, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

has contractible source. It therefore provides a point of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ (up to "contractible" ambiguity, which we shall ignore), which we shall refer to as the zero map and shall denote also by 0.

Let $\mathcal{C}$ be an ∞-category with a zero object 0. Given a morphism $g : Y \to Z$ in $\mathcal{C}$, a *kernel* for g is a fiber product $Y \times_Z 0$. Dually, a *cokernel* for g is a pushout $Z \coprod_Y 0$.

Definition 2.1.4. Let $\mathcal{C}$ be an ∞-category with a zero object. A *triangle* in $\mathcal{C}$ consists of a composable pair of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, together with a homotopy between $g \circ f$ and 0 in $\mathrm{Hom}_{\mathcal{C}}(X, Z)$.

Suppose that $g : Y \to Z$ is fixed. Completing this data to a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z$ is equivalent to providing a morphism from X into the kernel of g. We shall say that this triangle is *exact*, or a *fiber sequence*, if this map is an equivalence (so that X is a kernel for g). Dually, we shall say that a triangle is *co-exact*, or a *cofiber sequence*, if it exhibits Z as a cokernel for f.

Definition 2.1.5. An ∞-category $\mathcal{C}$ is *stable* if it satisfies the following conditions:

- The ∞-category $\mathcal{C}$ has a zero object.
- Every morphism in $\mathcal{C}$ has a kernel and a cokernel.
- A triangle in $\mathcal{C}$ is exact if and only if it is co-exact.

Example 2.1.6. Recall that a *spectrum* is a sequence $\{X_i\}$ of spaces equipped with base point, together with equivalences $X_i \simeq \Omega X_{i+1}$, where Ω denotes the loop space functor. The ∞-category of spectra is stable (the first two axioms follow formally, while the third may be deduced from the "homotopy excision theorem"). This is the motivation for our terminology: a stable ∞-category is an ∞-category which resembles the ∞-category of stable homotopy theory.

Remark 2.1.7. The third clause of Definition 2.1.5 is analogous to the axiom for abelian categories which asserts that the image of a morphism be isomorphic to its coimage.

Remark 2.1.8. One attractive feature of the notion of a stable ∞-category is that stability is a property of ∞-categories, rather than additional data which must be specified. We recall that a similar situation exists for additive categories. Although additive categories are usually presented as categories equipped with additional structure (an abelian group structure on all Hom-sets), this additional structure is in fact determined by the underlying category structure. If a category $\mathcal{C}$ has a zero object, finite sums, and finite products, then there always exists a unique map $A \oplus B \to A \times B$ which is given by the matrix

$$\begin{bmatrix} \mathrm{id}_A & 0 \\ 0 & \mathrm{id}_B \end{bmatrix}.$$

If this morphism has an inverse $\phi_{A,B}$, then we may define a sum of two morphisms $f, g : X \to Y$ by defining $f + g$ to be the composite $X \to X \times X \stackrel{f,g}{\to} Y \times Y \stackrel{\phi_{Y,Y}}{\to} Y \oplus Y \to Y$. In the presence of an additional assumption guaranteeing the existence of additive inverses, one may deduce that $\mathcal{C}$ is an additive category.

Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor between stable ∞-categories. Suppose that F carries zero objects into zero objects. It follows immediately that F carries triangles into triangles. If, in addition, F carries exact triangles into exact triangles, then we shall say that F is *exact*. We will write $\mathcal{EF}(\mathcal{C}, \mathcal{C}')$ for the ∞-category of exact functors from $\mathcal{C}$ to $\mathcal{C}'$ (considered as a full subcategory of the ∞-category of *all* functors from $\mathcal{C}$ to $\mathcal{C}'$). The ∞-category $\mathcal{EF}(\mathcal{C}, \mathcal{C}')$ is itself stable. Moreover, it is easy to see that exactness admits the following alternative characterizations:

Proposition 2.1.9. *Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor between stable ∞-categories. The following are equivalent:*

1. *The functor F is left exact. That is, F commutes with finite limits.*
2. *The functor F is right exact. That is, F commutes with finite colimits.*
3. *The functor F is exact.*

The identity functor from any stable ∞-category to itself is exact, and a composition of exact functors is exact. Consequently, we may consider the $(\infty, 2)$-category of stable ∞-categories and exact functors as a subcategory of the $(\infty, 2)$-category of all ∞-categories. We next note that this subcategory has good stability properties.

Example 2.1.10. If $\mathcal{C}$ is a stable ∞-category, then the opposite ∞-category $\mathcal{C}^{op}$ is also stable.

Example 2.1.11. If $\mathcal{C}$ is stable ∞-category, and $\mathcal{C}_0$ is a full subcategory containing a zero object and stable under the formation of kernels and cokernels, then $\mathcal{C}_0$ is stable and the inclusion $\mathcal{C}_0 \subseteq \mathcal{C}$ is exact.

Example 2.1.12. The $(\infty, 2)$-category of stable ∞-categories admits all $(\infty, 2)$-categorical limits, which are constructed by taking limits of the underlying ∞-categories.

Example 2.1.13. If $\mathcal{C}$ is a stable ∞-category and κ a regular cardinal, then $\mathrm{Ind}_\kappa(\mathcal{C})$ is a stable ∞-category.

Proposition 2.1.14. *If $\mathcal{C}$ is a stable ∞-category, then $h\,\mathcal{C}$ has the structure of a triangulated category.*

Proof. Let $X \in \mathcal{C}$ be an object. Then the zero map $X \to 0$ has a cokernel, which we shall write as $X[1]$. We shall call $X[1]$ the *suspension* of X. Similarly, we may define $X[-1]$ as the kernel of $0 \to X$; we call $X[-1]$ the *desuspension* of X. In view of the equivalence between exactness and co-exactness, we see that suspension and desuspension are naturally inverse to one another. Passing to the homotopy category, we obtain inverse equivalences which give rise to the shift functor on $h\,\mathcal{C}$.

We next prove that $h\,\mathcal{C}$ admits (finite) direct sums. It will suffice to prove that $\mathcal{C}$ itself admits finite direct sums, since any direct sum in $\mathcal{C}$ is also a direct sum in the homotopy category. We note that if $A' \to A \to A''$ and $B' \to B \to B''$ are exact triangles, $A' \oplus B'$ exists, and $A \oplus B$ exists, then $A'' \oplus B''$ may be constructed as the cokernel of the induced map $A' \oplus B' \to A \oplus B$. Now, for any pair of objects $X, Y \in \mathcal{C}$, we have exact triangles $0 \to X \to X$ and $Y[-1] \to 0 \to Y$. Thus, in order to construct $X \oplus Y$, it suffices to construct $0 \oplus Y[-1] \simeq Y[-1]$ and $X \oplus 0 \simeq X$.

To prove the additivity of $h\,\mathcal{C}$, we could proceed using the suggestion of Remark 2.1.8 to show that finite sums coincide with finite products. However, it will be easier (and more informative) to construct the additive structure directly. For any objects $X, Y \in \mathcal{C}$, we have $X \simeq 0 \times_{X[1]} 0$, so that $\mathrm{Hom}_{\mathcal{C}}(Y, X)$ is the loop space of $\mathrm{Hom}_{\mathcal{C}}(Y, X[1])$ (with respect to the base point given by the zero map). Iterating this construction, we can produce (functorially) arbitrarily many deloopings of $\mathrm{Hom}_{\mathcal{C}}(Y, X)$. In other words, the ∞-category $\mathcal{C}$ is naturally *enriched over spectra*, in the sense that for all $X, Y \in \mathcal{C}$ the space $\mathrm{Hom}_{\mathcal{C}}(Y, X)$ is the zeroth space of an associated spectrum which, by abuse of notation, we shall also denote by $\mathrm{Hom}_{\mathcal{C}}(Y, X)$. In particular, we note that $\mathrm{Hom}_{h\,\mathcal{C}}(Y, X) = \pi_0 \mathrm{Hom}_{\mathcal{C}}(Y, X) \simeq \pi_2 \mathrm{Hom}_{\mathcal{C}}(Y, X[2])$ has an abelian group structure, which is functorial in X and Y.

Now suppose that we are given an exact triangle in $\mathcal{C}$, consisting of a pair of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ and a nullhomotopy α of $g \circ f$. From this data we may construct a morphism $h : Z \to X[1]$, well defined up to homotopy, as follows. To give a map $Z \to X[1]$, we must give a map $h' : Y \to X[1]$ together with a nullhomotopy β of the composite $X \to X[1]$. We take h' to be the zero map, and β to be the tautological self-homotopy of $0 \in \mathrm{Hom}_{\mathcal{C}}(X, X[1])$. The pair (h', β) determines a map $h : Z \to X[1]$. We shall declare that the distinguished triangles in $h\,\mathcal{C}$ are precisely those which are isomorphic to those diagrams $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ which arise in this fashion.

This completes the construction of the triangulated structure on $h\,\mathcal{C}$. To finish the proof, one must verify that $h\,\mathcal{C}$ satisfies the axioms for a triangulated category. The details are somewhat tedious; we refer the reader to [15] for a proof in a related context. $\square$

Remark 2.1.15. We note that the definition of a stable ∞-category is quite a bit simpler than that of a triangulated category. In particular, the octahedral axiom is a consequence of ∞-categorical principles which are basic and easily motivated.

Remark 2.1.16. As noted in the proof of Proposition 2.1.14, any stable ∞-category $\mathcal{C}$ is naturally enriched over spectra. We will abuse notation by writing $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ to represent both the space of maps from X to Y *and* the corresponding spectrum. Since the former is simply the zeroth space of the latter, there is little risk of confusion: it will generally be clear from context whether we refer to the space or to the spectrum.

Proposition 2.1.17. *Let $\mathcal{C}$ be an ∞-category with a zero object. The following conditions are equivalent:*

1. *The ∞-category $\mathcal{C}$ is stable.*

2. *The ∞-category $\mathcal{C}$ has finite colimits and the suspension functor $X \mapsto 0 \coprod_X 0$ is an equivalence.*

3. *The ∞-category $\mathcal{C}$ has finite limits and the loop space functor $X \mapsto 0 \times_X 0$ is an equivalence.*

Proof. We show that (1) $\Leftrightarrow$ (2); the dual argument then gives (1) $\Leftrightarrow$ (3). To begin, suppose that $\mathcal{C}$ is stable. To show that $\mathcal{C}$ has all finite colimits, it suffices to show that $\mathcal{C}$ has an initial object, pairwise sums, and coequalizers. The existence of an initial object is clear from the definition, and the coequalizer of a pair $f, g : X \to Y$ may be constructed as the cokernel of the difference $f - g : X \to Y$ (which is well-defined up to homotopy, using the additive structure on $h\,\mathcal{C}$). The construction of sums was explained in the proof of Proposition 2.1.14. Finally, we note that a triangle $X \to 0 \to Y$ identifies Y with the suspension of X if and only if it identifies X with the loop space of Y; therefore the loop space functor is homotopy inverse to the suspension functor.

For the reverse direction, we sketch an argument which we learned from Bertrand Toën. Suppose that $\mathcal{C}$ has finite colimits and that the suspension functor is invertible. The invertibility of the suspension functor shows in particular that $\mathrm{Hom}_{\mathcal{C}}(X, Y) = \Omega^n \mathrm{Hom}_{\mathcal{C}}(X[-n], Y)$. We may therefore view $\mathcal{C}$ as an ∞-category which is *enriched over* the ∞-category $\mathcal{S}_\infty$ of spectra. By general nonsense, we obtain an "enriched Yoneda embedding" $\mathcal{C} \to \mathcal{D} = \mathcal{S}_\infty^{\mathcal{C}^{op}}$. Like the usual Yoneda embedding, this functor is fully faithful and we may therefore identify $\mathcal{C}$ which its essential image in $\mathcal{D}$.

The ∞-category $\mathcal{D}$ may be viewed as a limit of copies of $\mathcal{S}_\infty$. Consequently, $\mathcal{D}$ is stable. To prove that $\mathcal{C}$ is stable, it suffices to show that $\mathcal{C} \subseteq \mathcal{D}$ contains the zero object and is stable under the formation of kernels and cokernels. Stability under the formation of kernels is obvious from the definition, since the Yoneda embedding $\mathcal{C} \to \mathcal{D}$ commutes with all limits. Suppose $f : X \to Y$ is a morphism in $\mathcal{C}$, having a cokernel $Z \in \mathcal{D}$. We wish to prove that $Z \in \mathcal{C}$. Since $Z[-1]$ may be identified with the kernel of f, we deduce that $Z[-1]$ belongs to $\mathcal{C}$. Let Z' denote the suspension of $Z[-1]$ in $\mathcal{C}$. Since $Z[-1]$ is the loop space of Z' in $\mathcal{C}$, it is the loop space of Z' in $\mathcal{D}$, so that Z' is the suspension of $Z[-1]$ in $\mathcal{D}$. It follows that Z' is equivalent to Z, and belongs to $\mathcal{C}$. $\square$

We note that in a stable ∞-category, one can often reduce the consideration of general colimits to the simpler case of direct sums:

Proposition 2.1.18. *Let κ be a regular cardinal.*

- *A stable ∞-category $\mathcal{C}$ has all κ-small colimits if and only if $\mathcal{C}$ has all κ-small filtered colimits, if and only if $\mathcal{C}$ has all κ-small sums.*

- *An exact functor $F : \mathcal{C} \to \mathcal{C}'$ between stable ∞-categories preserves κ-small colimits if and only if F preserves all κ-small filtered colimits, if and only if F preserves all κ-small sums.*

- *An object $X \in \mathcal{C}$ is κ-compact if and only if any map $X \to \oplus_{\alpha \in A} Y_\alpha$ factors through $\oplus_{\alpha \in A_0} Y_\alpha$, for some $A_0 \subseteq A$ having size $< \kappa$.*

Corollary 2.1.19. *Let $\mathcal{C}$ be a stable ∞-category. Then $\mathcal{C}$ is presentable if and only if it satisfies the following conditions:*

- *The ∞-category $\mathcal{C}$ admits arbitrary sums.*

- *There exists a small generator for $\mathcal{C}$. That is, there exists an object $X \in \mathcal{C}$ and a cardinal κ such that $\mathrm{Hom}_{\mathcal{C}}(X, Y) = 0$ implies $Y = 0$, and any map $X \to \oplus_{\alpha \in A} Y_\alpha$ factors through $\oplus_{\alpha \in A_0} Y_\alpha$ for some subset $A_0 \subseteq A$ of size $< \kappa$.*

Remark 2.1.20. If $\{X_\alpha\}_{\alpha \in A}$ is a family of objects of a stable ∞-category $\mathcal{C}$, each equipped with a map to $X \in \mathcal{C}$, then these maps exhibit X as the direct sum of $\{X_\alpha\}_{\alpha \in A}$ if and only if X is a direct sum in the homotopy category $h\,\mathcal{C}$. Consequently, the presentability of $\mathcal{C}$ is equivalent to a set of conditions on the homotopy category $h\,\mathcal{C}$ which may be studied independently of the assumption that $h\,\mathcal{C}$ is the homotopy category of a stable ∞-category: see [29].

2.2 Localizations of Stable ∞-Categories

Let $\mathcal{C}$ be a triangulated category. We recall that a *t-structure* on $\mathcal{C}$ is defined to be a pair of full subcategories $\mathcal{C}_{\geq 0}$, $\mathcal{C}_{\leq 0}$ (always assumed to be stable under isomorphism) having the following properties:

- For $X \in \mathcal{C}_{\geq 0}$ and $Y \in \mathcal{C}_{\leq 0}[-1]$, we have $\mathrm{Hom}_{\mathcal{C}}(X, Y) = 0$.

- $\mathcal{C}_{\geq 0}[1] \subseteq \mathcal{C}_{\geq 0}$, $\mathcal{C}_{\leq 0}[-1] \subseteq \mathcal{C}_{\leq 0}$.

- For any $X \in \mathcal{C}$, there exists a distinguished triangle $X' \to X \to X'' \to X'[1]$ where $X' \in \mathcal{C}_{\geq 0}$ and $X'' \in \mathcal{C}_{\leq 0}[-1]$.

We write $\mathcal{C}_{\geq n}$ for $\mathcal{C}_{\geq 0}[n]$ and $\mathcal{C}_{\leq n}$ for $\mathcal{C}_{\leq 0}[n]$, and we let $\mathcal{C}_{\geq n, \leq m} = \mathcal{C}_{\geq n} \cap \mathcal{C}_{\leq m}$. We refer the reader to [5] for a more detailed discussion of t-structures on triangulated categories (though our notation differs slightly from theirs, since we use a "homological" rather than "cohomological" indexing).

If $\mathcal{C}$ is a stable ∞-category, then a *t-structure* on $\mathcal{C}$ is defined to be a t-structure on its homotopy category $h\,\mathcal{C}$ (which is triangulated by Proposition 2.1.14). In this case, for any $X \in \mathcal{C}$ and any $k \in \mathbf{Z}$, one has an exact triangle $\tau_{\geq k}X \to X \to \tau_{\leq k-1}X$, where $\tau_{\geq k}X \in \mathcal{C}_{\geq k}$ and $\tau_{\leq k-1}X \in \mathcal{C}_{\leq k-1}$. This triangle is unique up to a contractible space of choices, so that $\tau_{\geq k}$ and $\tau_{\leq k}$ may be regarded as functors $\mathcal{C} \to \mathcal{C}$. One also checks that there is a natural equivalence between the composite functors $\tau_{\geq k}\tau_{\leq n}$ and $\tau_{\leq n}\tau_{\geq k}$; this composite functor will be denoted by $\tau_{\geq k, \leq n}$. In particular, we let $\pi_k = \tau_{\geq k, \leq k}[-k]$, so that π_k^t maps $\mathcal{C}$ into $\mathcal{C}_0 = \mathcal{C}_{\leq 0, \geq 0}$. We note that $\mathcal{C}_0$ is an ordinary category, and therefore equivalent to $h\,\mathcal{C}_0 \subseteq h\,\mathcal{C}$. This subcategory of $h\,\mathcal{C}$ is called the *heart* of $h\,\mathcal{C}$ and is abelian (see [5]). Consequently, we shall refer to $\mathcal{C}_0$ as the *heart* of $\mathcal{C}$.

Example 2.2.1. Let $\mathcal{S}_\infty$ denote the ∞-category of spectra. Then $\mathcal{S}_\infty$ has a t-structure, given by $(\mathcal{S}_\infty)_{\geq 0} = \{X \in \mathcal{S}_\infty : (\forall i < 0)[\pi_i X = 0]\}$, $(\mathcal{S}_\infty)_{\leq 0} = \{X \in \mathcal{S}_\infty : (\forall i > 0)[\pi_i X = 0]\}$. The heart of $\mathcal{S}_\infty$ is equivalent to the category $\mathcal{A}$ of abelian groups. The functor $\pi_k : \mathcal{S}_\infty \to \mathcal{A}$ defined above agrees with the usual functor π_k, which assigns to a spectrum X the group $[S^k, X]$ of homotopy classes of maps from a k-sphere into X.

For the remainder of this section, we shall discuss the relationship between t-structures and Bousfield localizations of stable ∞-categories. For a discussion of Bousfield localization from our point of view, we refer the reader to [22]. We briefly summarize the theory here. Given a presentable ∞-category $\mathcal{C}$ and a *set* $S = \{f : X \to Y\}$ of morphisms of $\mathcal{C}$, we shall say that an object $Z \in \mathcal{C}$ is S*-local* if, for any $f : X \to Y$ belonging to S, the induced map of spaces $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ is an equivalence. The basic result of the theory asserts that for any $X \in \mathcal{C}$, there exists a map $\phi : X \to LX$ where LX is S-local, and the morphism ϕ is, in some sense, built out of the morphisms of S (in the language of [22], ϕ belongs to the saturated class generated by S). Moreover, the morphism $\phi : X \to LX$ is essentially unique, and functorially determined by X.

Let us now suppose that $\mathcal{C}$ is a stable ∞-category. An object $Z \in \mathcal{C}$ is S-local if and only if any map $f : X \to Y$ in S induces a homotopy equivalence of *spaces* $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$, which means that $\phi_i : \pi_i \mathrm{Hom}_{\mathcal{C}}(Y, Z) \simeq \pi_i \mathrm{Hom}_{\mathcal{C}}(X, Z)$ for $i \geq 0$. These morphisms of homotopy groups fit into a long exact sequence

$$\ldots \to \pi_i \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \pi_i \mathrm{Hom}_{\mathcal{C}}(X, Z) \to \pi_i \mathrm{Hom}_{\mathcal{C}}(\ker f, Z) \to \pi_{i-1} \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \ldots$$

From this long exact sequence, we see that if Z is S-local, then $\pi_i \mathrm{Hom}_{\mathcal{C}}(\ker f, Z)$ vanishes for $i > 0$. Conversely, if $\pi_i \mathrm{Hom}_{\mathcal{C}}(\ker f, Z)$ vanishes for $i \geq 0$, then Z is S-local. Experience teaches us that in situations such as this, vanishing conditions on the homotopy groups $\pi_i \mathrm{Hom}_{\mathcal{C}}(\ker f, Z)$ are more natural than conditions which assert the invertibility of the homomorphisms ϕ_i. It is therefore natural to wonder if the S-locality of Z is *equivalent* to the vanishing of certain homotopy groups $\pi_i \mathrm{Hom}_{\mathcal{C}}(\ker f, Z)$. This is not true in general, but may instead be taken as a characterization of a good class of localizations of $\mathcal{C}$:

Proposition 2.2.2. *Let $\mathcal{C}$ be a stable, presentable ∞-category. Let $L : \mathcal{C} \to \mathcal{C}$ be a localization functor, and let S be the collection of all morphisms f in $\mathcal{C}$ for which Lf is an equivalence. The following conditions are equivalent:*

1. *The class S is generated (as a saturated collection of morphisms) by a set of morphisms of the form $0 \to A$, $A \in \mathcal{C}$.*

2. *The class S is generated by the morphisms $\{0 \to A : LA \simeq 0\}$.*

3. *If $X \to Y \to Z$ is a fiber sequence, where X and Z are S-local, then Y is also S-local.*

4. *For any $A \in \mathcal{C}$, $B \in L\,\mathcal{C}$, the natural map $\pi_{-1} \operatorname{Hom}(LA, B) \to \pi_{-1} \operatorname{Hom}(A, B)$ is injective.*

5. *The full subcategories $\mathcal{C}_{\geq 0} = \{A : LA \simeq 0\}$and $\mathcal{C}_{\leq -1} = \{A : LA \simeq A\}$ determine a t-structure on $\mathcal{C}$.*

Proof. We will show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1). It is clear that (1) implies (2). Assuming (2), we see that an object Y is S-local if and only if $\pi_0 \operatorname{Hom}_{\mathcal{C}}(A, Y) = 0$ for any A such that $LA = 0$. Then (3) follows from the long exact sequence.

Assume (3), let $B \in L\,\mathcal{C}$, and let $\eta \in \pi_{-1} \operatorname{Hom}_{\mathcal{C}}(LA, B)$ classify an extension $B \to C \to LA \to B[1]$. Condition (4) implies that C is S-local. If the image of η in $\pi_{-1} \operatorname{Hom}_{\mathcal{C}}(A, B)$ is trivial, then the induced extension of A is split by some map $A \to C$. Since C is S-local, this map factors through LA, so that $\eta = 0$. This proves (4).

Assume (4), and define $\mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq -1}$ as in (5). If $X \in \mathcal{C}_{\geq 0}$ and $Y \in \mathcal{C}_{\leq -1}$, then $\operatorname{Hom}_{\mathcal{C}}(X, Y) = \operatorname{Hom}_{\mathcal{C}}(LX, Y) = \operatorname{Hom}_{\mathcal{C}}(0, Y) = 0$. Stability of $\mathcal{C}_{\leq -1}$ under the loop space functor follows from a general fact that local objects are closed under the formation of limits. The stability of $\mathcal{C}_{\geq 0}$ under suspensions follows from the fact that $L : \mathcal{C} \to L\,\mathcal{C}$ commutes with colimits. To complete the verification of (5), we consider for any $X \in \mathcal{C}$ the triangle

$$X' \to X \to LX \to X'[1].$$

It will suffice to show that $LX' = 0$, or that $\operatorname{Hom}_{\mathcal{C}}(X', Y) = 0$ for any Y which is S-local. Replacing Y by a suitable loop space of Y, it suffices to show that $0 = \pi_0 \operatorname{Hom}_{\mathcal{C}}(X', Y) = \pi_{-1} \operatorname{Hom}_{\mathcal{C}}(X'[1], Y)$. This follows from (4) and the long exact sequence.

Finally, suppose that (5) is satisfied. A cardinality argument shows that the collection of all objects $A \in \mathcal{C}$ such that $LA = 0$ is generated under filtered colimits by some set of objects $\mathfrak{S}$. Let L' be the localization functor which inverts every morphism $\{0 \to A : A \in \mathfrak{S}\}$; we wish to show that $L \simeq L'$. Clearly L factors through L'; replacing $\mathcal{C}$ by $L'\,\mathcal{C}$, we may assume that L' is the identity so L does not kill any nonzero objects. In this case, $\mathcal{C}_{\geq 0}$ consists only of zero. Now (5) implies that $\mathcal{C}_{\leq -1} = \mathcal{C}$. □

We will call a t-structure on a stable, presentable ∞-category *admissible* if it arises from a localization satisfying the equivalent conditions of Proposition 2.2.2.

Proposition 2.2.3. *Let $\mathcal{C}$ be an ∞-category. The following conditions are equivalent:*

1. *There exists a presentable, stable ∞-category $\mathcal{C}'$, an admissible t-structure on $\mathcal{C}'$, and an equivalence $\mathcal{C} \simeq (\mathcal{C}')_{\geq 0}$.*

2. *The ∞-category $\mathcal{C}$ is presentable, has a zero object, and the suspension functor S is fully faithful.*

Proof. It is easy to see that (1) implies (2). For the converse, we take $\mathcal{C}'$ to be the ∞-category of "infinite loop" objects of $\mathcal{C}$. Namely, let $\mathcal{C}'$ denote the limit of the inverse system

$$\ldots \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C}$$

of ∞-categories. In other words, an object of $\mathcal{C}$ is given by a sequence $\{X_i\}$ of objects of $\mathcal{C}$ together with equivalences $f_i : X_i \simeq \Omega X_{i+1}$. Here $\Omega : \mathcal{C} \to \mathcal{C}$ denotes the loop space functor, given by $\Omega X = 0 \times_X 0$.

We note that the construction of $\mathcal{C}'$ from $\mathcal{C}$ is precisely analogous to the construction of the ∞-category $\mathcal{S}_\infty$ of spectra from the ∞-category $\mathcal{S}_*$ of pointed spaces. We will therefore borrow terminology from the theory of spectra, and speak of X_i as the *ith space* of an object $\{X_i, f_i\} \in \mathcal{C}'$. In particular, we write $\Omega^\infty\{X_i, f_i\} = X_0$ so that $\Omega^\infty : \mathcal{C}' \to \mathcal{C}$ is the *zeroth space* functor. The functor Ω^∞ has a left adjoint S^∞, which carries an object $X \in \mathcal{C}$ to the "suspension spectrum" given by the sequence $X, SX, S^2X, \ldots$. Here S denotes the suspension functor $Y \mapsto 0 \coprod_Y 0$, and we make use of the canonical equivalences $\Omega S^{n+1}X \simeq S^n X$ which result from the hypothesis that the suspension functor is fully faithful.

Since $\mathcal{C}'$ is a limit of copies of $\mathcal{C}$, it is presentable. By construction, the suspension functor of $\mathcal{C}'$ is invertible so that $\mathcal{C}'$ is stable. We can endow $\mathcal{C}'$ with the t-structure "generated by" the objects of the form $S^\infty X$. Namely, we consider the admissible t-structure corresponding to the localization which inverts every morphism $0 \to S^\infty X$, $X \in \mathcal{C}$ (the presentability of $\mathcal{C}$ implies that it suffices to kill a set of objects X which generate $\mathcal{C}$ under colimits).

We first prove that for any $X \in \mathcal{C}$, the adjunction morphism $S\Omega X \to X$ is a monomorphism. Since S^∞ is a fully faithful embedding of $\mathcal{C}$ into $\mathcal{C}'$, we see that $\mathcal{C}$ is enriched over spectra, so that it suffices to show that the kernel of $S\Omega X \to X$ is zero. For this, we just need to know that the induced map $\Omega S\Omega X \to \Omega X$ is an equivalence. A homotopy inverse is given by the adjunction map $Y \to \Omega SY$, where $Y = \Omega X$.

Now S^∞ is a fully faithful embedding of $\mathcal{C}$ into $\mathcal{C}'$, and by construction it factors through $\mathcal{C}'_{\geq 0}$. Since S^∞ is left adjoint to the "zeroth space" functor Ω^∞, we see that $\mathcal{C}'_{<0} = \{X \in \mathcal{C}' : \Omega^\infty X = 0\}$. To complete the proof, it suffices to show that if $X = (\ldots, X_1, X_0) \in \mathcal{C}'$ and $\mathrm{Hom}_{\mathcal{C}'}(X, Y) = 0$ whenever $\Omega^\infty Y = 0$, then X is (equivalent to) a suspension spectrum. In other words, we need to show that the natural map $f : S^\infty X_0 \to X$ is an equivalence. Let K be the cokernel of f. Since $\mathcal{C}_{\geq 0}$ is stable under colimits, we deduce that $K \in \mathcal{C}_{\geq 0}$. If $\Omega^\infty K = 0$, then the identity map from K to itself is nullhomotopic, so that $K = 0$ and f is an equivalence.

Now, $K[-1]$ is the kernel of f. The 0th space of K is equivalent to the 1st space of $K[-1]$, which is the kernel of the natural map $SX_0 \to X_1$. But, as we noted above, the adjunction morphism $S\Omega X_1 \to X_1$ is a monomorphism, so its kernel vanishes. This completes the proof. □

Remark 2.2.4. Of course, the ∞-category $\mathcal{C}'$ of Proposition 2.2.3 is not unique. However, the candidate constructed in the proof is the unique choice which is *right-complete* with respect to its t-structure, in terminology introduced below.

Remark 2.2.5. Let $\mathcal{C}$ be a presentable stable ∞-category, and let $\mathcal{C}_0$ be a localization of $\mathcal{C}$. Let $L : \mathcal{C} \to \mathcal{C}_0$ denote the localization functor. Then $\mathcal{C}_0$ is stable if and only if L is left exact. In particular, since $\mathcal{C}$ is easily seen to be a localization of the ∞-category of presheaves of spectra on some subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$, we see that the class of stable, presentable ∞-categories may be characterized as the smallest class of ∞-categories which contains the ∞-category $\mathcal{S}_\infty$ of spectra and is stable under limits and left exact localizations. This result is analogous to Giraud's characterization of topoi (see [3]), its ∞-categorical analogue (see [22]), and the Gabriel-Popesco theorem for abelian categories (see [28]).

We next discuss various boundedness notions which can be associated to t-structures. Assume that $\mathcal{C}$ is a stable ∞-category equipped with a t-structure. Let $\mathcal{C}^+ = \bigcup_i \mathcal{C}_{\leq i}$. We shall say that $\mathcal{C}$ is *left bounded* if $\mathcal{C}^+ = \mathcal{C}$.

At the other extreme, given a stable ∞-category $\mathcal{C}$ equipped with a t-structure, we define the *left completion* $\hat{\mathcal{C}}$ of $\mathcal{C}$ to be the limit of the following tower of ∞-categories:

$$\ldots \to \mathcal{C}_{\leq 2} \stackrel{\tau_{\leq 1}}{\to} \mathcal{C}_{\leq 1} \stackrel{\tau_{\leq 0}}{\to} \mathcal{C}_{\leq 0}$$

In other words, an object of $\hat{\mathcal{C}}$ is a sequence $\{C_i\}_{i \geq 0}$ together with equivalences $C_i \simeq \tau_{\leq i} C_{i+1}$ (which imply that $C_i \in \mathcal{C}_{\leq i}$). There is a natural functor $\mathcal{C} \to \hat{\mathcal{C}}$ which carries an object C to the sequence $\{\tau_{\leq i} C\}$. We shall say that $\mathcal{C}$ is *left complete* if this functor is an equivalence.

Proposition 2.2.6. *For any stable ∞-category $\mathcal{C}$ equipped with a t-structure, the ∞-categories $\mathcal{C}^+$ and $\hat{\mathcal{C}}$ are stable. The functors $\mathcal{C}^+ \to \mathcal{C} \to \hat{\mathcal{C}}$ are exact. Moreover, there are natural equivalences $\hat{\mathcal{C}} \simeq \widehat{\mathcal{C}^+}$ and $(\hat{\mathcal{C}})^+ \simeq \mathcal{C}^+$.*

Consequently, we see that the concepts of "left bounded" and "left complete" stable ∞-categories (with t-structure) are essentially interchangeable.

The following proposition gives a good criterion for detecting left completeness:

Proposition 2.2.7. *Suppose that $\mathcal{C}$ is a stable ∞-category with t-structure, which admits countable products. Suppose further that $\mathcal{C}_{\geq 0}$ is stable under countable products. Then $\mathcal{C}$ is left complete if and only if any $A \in \mathcal{C}_{\geq 0}$ such that $\pi_k A = 0$ for $k \geq 0$ is itself zero.*

Proof. Since $\mathcal{C}_{\leq 0}$ is stable under all limits, the assumption implies that $\mathcal{C}_0$ is stable under countable products. Since products of exact triangles are exact, the existence of the triangles

$$\tau_{\geq 0} X \to X \to \tau_{<0} X$$

$$\tau_{\geq 1} X \to \tau_{\geq 0} X \to \pi_0 X,$$

implies that the formation of homotopy groups is compatible with countable products.

The necessity is obvious; let us therefore assume that any element of $\mathcal{C}_{\geq 0}$ whose homotopy groups vanish is identically zero. It follows that if $f : A \to B$ is a morphism of objects of $\mathcal{C}_{\geq k}$ which induces an isomorphism on homotopy, then f is an equivalence.

Since $\mathcal{C}$ admits countable products, it admits countable limits (by the dual form of Proposition 2.1.18). Consequently, the natural functor $F : \mathcal{C} \to \hat{\mathcal{C}}$ has a right adjoint G, given by formation of the inverse limit. We first claim that the natural map $A \to GFA$ is an equivalence for each $A \in \mathcal{C}$. It is clear that G and F are the identity on $\mathcal{C}^+ \subseteq \mathcal{C}, \hat{\mathcal{C}}$. Since F and G are both exact, we may reduce to the case where $A \in \mathcal{C}_{\geq 1}$.

Now $GFA = \lim_n \tau_{\leq n} A$. This limit may be constructed as the kernel of a map from $\prod \tau_{\leq n} A$ to itself. Consequently, we deduce that GFA lies in $\mathcal{C}_{\geq 0}$. Splicing the long exact sequence of the associated triangle, we deduce the existence of short exact sequences

$$0 \to \lim{}^1\{\pi_{k+1}\tau_{\leq n}A\} \to \pi_k GFA \to \lim{}^0\{\pi_k \tau_{\leq n} A\} \to 0$$

in the abelian category $\mathcal{C}_0$. Since both of the inverse systems in question are eventually constant, we get $\pi_k GFA \simeq \pi_k A$. Thus, the cokernel of $A \to GFA$ lies in $\mathcal{C}_{\geq 0}$ and has vanishing homotopy groups, and therefore is itself zero by the hypothesis.

To complete the proof, we wish to show that the natural map $FGA \to A$ is an equivalence for any $A = \{A_n\} \in \hat{\mathcal{C}}$. It suffices to treat the cases $A \in \hat{\mathcal{C}}_{\leq 0}$ and $A \in \hat{\mathcal{C}}_{\geq 1}$ separately. The first case is obvious, since the sequence $\{A_n\}$ is eventually constant. In the second case, we note that the above calculation shows that $\pi_n GA \simeq \pi_n A_k$ for $k \geq n$, so that the natural map $\tau_{\leq n} GA \to A_n$ induces an isomorphism on homotopy groups. Using the hypothesis, we conclude that $\tau_{\leq n} GA \simeq A_n$ so that $FGA \simeq A$ as required. □

The preceding notions may all be dualized. We thus obtain notions of *right bounded* and *right complete* stable ∞-category (with t-structure). We denote the subcategory $\bigcup_n \mathcal{C}_{\geq -n}$ by $\mathcal{C}^-$.

Remark 2.2.8. The notions of right and left boundedness (or completeness) are essentially independent of one another. For example, the constructions introduced above for forming left completions and left bounded subcategories *commute* with the analogous constructions on the right. In particular we have $(\mathcal{C}^-)^+ = (\mathcal{C}^+)^- = \mathcal{C}^+ \cap \mathcal{C}^- = \mathcal{C}^b$, the stable subcategory of *t-bounded objects* of $\mathcal{C}$.

2.3 Abelian Categories

Some of the most important examples of stable ∞-categories are given by (some variant of) chain complexes in an abelian category. In this section, we will review Verdier's theory of derived categories from the ∞-categorical point of view.

Throughout this section, we shall restrict our attention to *Grothendieck abelian categories.* Recall that an abelian category is *Grothendieck* if it admits filtered colimits which are exact, and has a small generator. In other words, a Grothendieck abelian category is a presentable abelian category in which the class of monomorphisms is stable under filtered colimits.

If $\mathcal{A}$ is a Grothendieck abelian category, then we shall call a complex $K_\bullet$ in $\mathcal{A}$ *injective* if it has the following property: for any inclusion of complexes $i : M_\bullet \subseteq N_\bullet$ and any map of complexes $\phi : M_\bullet \to K_\bullet$, the map ϕ extends to a map $N_\bullet \to K_\bullet$ provided that i is a quasi-isomorphism. If $K_\bullet$ is injective, then each K_n is an injective object of $\mathcal{A}$. The converse holds provided that $K_\bullet$ is *left bounded* in the sense that $K_n = 0$ for $n \gg 0$, but not in general. The notion of an injective complex was introduced by Spaltenstein (see [32]), who showed there are "enough" injective complexes in the sense that any complex $K_\bullet$ admits a quasi-isomorphic inclusion into an injective complex. His work has a natural interpretation in the language of model categories: one may equip the category $\mathrm{Ch}(\mathcal{A})$ of chain complexes in $\mathcal{A}$ with a Quillen model structure in which the weak equivalences are quasi-isomorphisms and the cofibrations are chain maps that are termwise monic. In this case, a complex is fibrant if and only if it is injective in the sense described above.

If $M_\bullet$ and $N_\bullet$ are complexes in $\mathcal{A}$, then we may view $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(M_\bullet, N_\bullet)$ as a bicomplex of abelian groups. Passing to the associated complex and truncating, we obtain a complex of abelian groups which is concentrated in (homological) degrees ≥ 0. Via the Dold-Kan correspondence, we may view this as a simplicial abelian group. Viewing the underlying simplicial set as a space, we obtain a space which we may denote by $\mathrm{Hom}(M_\bullet, N_\bullet)$. We note that $\pi_n \mathrm{Hom}(M_\bullet, N_\bullet)$ is simply the group of chain-homotopy classes of maps from $M_\bullet$ to $N_{\bullet+n}$.

We define the *derived category* $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$ to be the ∞-category having as objects the injective complexes of $\mathcal{A}$, and as morphisms the spaces $\mathrm{Hom}(M_\bullet, N_\bullet)$ defined above. Equivalently, $\mathcal{D}(\mathcal{A})$ may be constructed as the simplicial localization of $\mathrm{Ch}(\mathcal{A})$, obtained by inverting class of all quasi-isomorphisms.

There is a natural $\mathcal{A}$-valued cohomological functor $\{\pi_i\}_{i\in\mathbf{Z}} : \mathcal{D}(\mathcal{A}) \to \mathcal{A}$. The functor π_i assigns to a complex $K_\bullet$ the group $\pi_i K_\bullet = \mathrm{H}^{-i}(K_\bullet)$. This functor induces a t-structure on $\mathcal{D}(\mathcal{A})$, which is given by setting $\mathcal{D}(\mathcal{A})_{\geq 0} = \{X \in \mathcal{D}(\mathcal{A}) : (\forall i < 0)[\pi_i X = 0]\}$ and $\mathcal{D}(\mathcal{A})_{\leq 0} = \{X \in \mathcal{D}(\mathcal{A}) : (\forall i > 0)[\pi_i X = 0]\}$

Remark 2.3.1. Our definition does not conform to the standard terminology, according to which it is actually the homotopy category $h\,\mathcal{D}(\mathcal{A})$ which is the derived category of $\mathcal{A}$. However, the shift in terminology seems appropriate since we will be much more concerned with $\mathcal{D}(\mathcal{A})$ than with its homotopy category.

Proposition 2.3.2. *Suppose that $\mathcal{A}$ is a Grothendieck abelian category. Then $\mathcal{D}(\mathcal{A})$ is left and right complete, and its heart is equivalent to $\mathcal{A}$. Furthermore, the formation of homotopy groups in $\mathcal{D}(\mathcal{A})$ is compatible with the formation of filtered colimits, so in particular $\mathcal{D}(\mathcal{A})_{\leq 0}$ is stable under filtered colimits.*

We note that $\mathcal{D}(\mathcal{A})$ may be obtained from the ordinary category of $\mathcal{A}$-valued complexes by a simplicial localization construction, which inverts all quasi-isomorphisms. A similar remark applies to the subcategory $\mathcal{D}^+(\mathcal{A})$: this is a simplicial localization of the ordinary category of chain complexes concentrated in (homological) degrees ≤ 0. Since any chain complex has an injective resolution, we could just as well consider only the ordinary category of $\mathcal{A}^{\text{inj}}$-valued chain complexes, where $\mathcal{A}^{\text{inj}} \subseteq \mathcal{A}$ denotes the full subcategory consisting of injective objects. By the Dold-Kan correspondence, this is equivalent to the category of cosimplicial objects of $\mathcal{A}^{\text{inj}}$. Moreover, from this point of view the simplicial localization has a very simple interpretation: it simply inverts morphisms between cosimplicial objects which admit a homotopy inverse (in the cosimplicial sense). This makes it very easy to describe $\mathcal{D}^+(\mathcal{A})$ in terms of a universal mapping property:

Proposition 2.3.3. *Let $\mathcal{A}$ be a Grothendieck abelian category, and let $\mathcal{C}$ be a stable ∞-category equipped with a right-complete t-structure. Let $\mathcal{F}$ denote the ∞-category of exact functors $F : \mathcal{D}^+(\mathcal{A}) \to \mathcal{C}$ which are left t-exact (that is, $F(\mathcal{D}(\mathcal{A})_{\leq 0}) \subseteq \mathcal{C}_{\leq 0}$) and carry injective objects of $\mathcal{A}$ into $\mathcal{C}_0$. Let $\mathcal{F}'$ denote the ordinary category of left exact functors $\mathcal{A} \to \mathcal{C}_0$. Then the restriction to the heart followed by truncation induces an equivalence $\mathcal{F} \to \mathcal{F}'$.*

Proof. We first note that $\mathcal{F}'$ is equivalent to the ordinary category of additive functors $\mathcal{A}^{\text{inj}} \to \mathcal{C}_0$, the equivalence being given by restriction of functors to $\mathcal{A}^{\text{inj}}$. To see this, we note that a left-exact functor $F : \mathcal{A} \to \mathcal{C}_0$ can be reconstructed (in an essentially unique way) from its restriction to $\mathcal{A}^{\text{inj}}$, since $F(X) \simeq \ker(F(I^0) \to F(I^1))$, where $I^\bullet$ is an injective resolution of X.

Essentially the same argument works to show that $\mathcal{F}$ is equivalent to the category of additive functors $\mathcal{A}^{\text{inj}} \to \mathcal{C}_0$. Once again, the equivalence is given by restriction. We will sketch the construction of a homotopy inverse to this equivalence. Suppose that we are given an additive functor $F : \mathcal{A}^{\text{inj}} \to \mathcal{C}_0$ as above. Let $\mathcal{A}^{\text{inj}}_\Delta$ denote the ordinary category of cosimplicial objects of $\mathcal{A}^{\text{inj}}$. Then applying F termwise gives a functor F_Δ from $\mathcal{A}^{\text{inj}}_\Delta$ to the ∞-category of cosimplicial objects of $\mathcal{C}_{\leq 0}$. Passing to the geometric realization, we obtain a functor $\widetilde{F'} : \mathcal{A}^{\text{inj}}_\Delta \to \mathcal{C}_{\leq 0}$.

Moreover, since quasi-isomorphisms in $\mathcal{A}^{\text{inj}}_\Delta$ admit simplicial homotopy inverses, one can easily check that $\widetilde{F'}$ carries quasi-isomorphisms into equivalences, so it induces a functor $F' : \mathcal{D}(\mathcal{A})_{\leq 0} \to \mathcal{C}_{\leq 0}$. Using the fact that F is left exact, one shows that F' is left exact, and therefore lifts uniquely to an exact functor $\mathcal{D}^+(\mathcal{A}) \to \mathcal{C}^+$ which we shall denote also by F'. By construction, F' is left t-exact. Moreover, if $I \in \mathcal{A}$ is injective, then it may be represented by the constant cosimplicial object with value I in $\mathcal{A}_0^\Delta$, which is carried by $\widetilde{F'}$ into $F(I)$. Thus $F'(I) = F(I) \in \mathcal{C}_0$. In general, F' is a right-derived functor of F and does not carry $\mathcal{A}$ into $\mathcal{C}_0$; however, one can check easily from the definition that $\tau_{\geq 0} F'(A) = F(A) \in \mathcal{C}_0$. □

Example 2.3.4. Any left-exact functor $G : \mathcal{A} \to \mathcal{A}'$ between Grothendieck abelian categories lifts naturally to a *right-derived functor* $RG : \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{A}')$. Often RG has a natural extension to all of $\mathcal{D}(\mathcal{A})$. For example, suppose that G is the right adjoint to some exact functor $F : \mathcal{A}' \to \mathcal{A}$. Then F induces a functor $LF : \mathcal{D}(\mathcal{A}') \to \mathcal{D}(\mathcal{A})$. The adjoint functor theorem can then be applied to construct a right adjoint to LF, which coincides with RG on $\mathcal{D}^+(\mathcal{A})$.

Example 2.3.5. Let $\mathcal{S}_\infty$ be the stable ∞-category of spectra, with its natural t-structure. Then the heart of $\mathcal{S}_\infty$ is the category $\mathcal{A}$ of abelian groups. The identity functor $\mathcal{A} \to \mathcal{A}$

lifts to a functor $\mathcal{D}(\mathcal{A}) \to \mathcal{S}_\infty$, which is given by taking a complex of abelian groups to the corresponding "generalized Eilenberg-MacLane spectrum".

Next, we ask under what conditions a stable ∞-category $\mathcal{C}$, equipped with a t-structure, has the form $\mathcal{D}(\mathcal{C}_0)$:

Proposition 2.3.6. *Let $\mathcal{C}$ be a presentable stable ∞-category equipped with an admissible t-structure.*

- *The heart $\mathcal{C}_0$ of $\mathcal{C}$ is a presentable abelian category.*
- *If $\mathcal{C}_{\leq 0}$ is stable under the formation of filtered colimits, then $\mathcal{C}_0$ is a Grothendieck abelian category.*
- *Suppose that $\mathcal{C}_{\leq 0}$ is stable under the formation of filtered colimits and that*

  $$\bigcap_{n \geq 0} \mathcal{C}_{\leq -n}$$

  contains only zero objects (so that $\mathcal{C}$ is right-complete). Then the functor $\mathcal{D}^+(\mathcal{C}_0) \to \mathcal{C}^+$ supplied by Proposition 2.3.3 is an equivalence if and only if $\pi_0 \operatorname{Hom}_{\mathcal{C}}(X, I[n]) = 0$ for each $X \in \mathcal{C}_0$, each injective object $I \in \mathcal{C}_0$, and each $n > 0$.

Proof. Since the t-structure is admissible, $\mathcal{C}_{\geq 0} \subseteq \mathcal{C}$ is generated under colimits by a set of objects and is therefore presentable. Since $\mathcal{C}_0$ is a localization of $\mathcal{C}_{\geq 0}$, it follows that $\mathcal{C}_0$ is presentable.

Let us now consider the question of whether or not $\mathcal{C}_0$ is a Grothendieck abelian category. Since $\mathcal{C}_0$ is presentable, we are interested in the condition that a filtered colimit of short exact sequences $0 \to A_\alpha \to B_\alpha \to C_\alpha \to 0$ remains exact. Let A, B, and C denote the corresponding filtered colimits in $\mathcal{C}$. Then $\pi_0 A$, $\pi_0 B$, and $\pi_0 C$ are the corresponding filtered colimits in $\mathcal{C}_0$. To prove the exactness of the sequence $0 \to \pi_0 A \to \pi_0 B \to \pi_0 C \to 0$, it suffices to prove that $\pi_1 C = 0$. This is certainly the case if $C \in \mathcal{C}_0$, which follows from the assumption that $\mathcal{C}_0$ is stable under filtered colimits.

Let us now consider the question of whether or not the functor $F : \mathcal{D}^+(\mathcal{C}_0) \to \mathcal{C}^+$ is fully faithful. The condition that $\pi_0 \operatorname{Hom}_{\mathcal{C}}(X, I[n]) = 0$ for $X \in \mathcal{C}_0$, $n > 0$ is equivalent to the assertion that $\operatorname{Hom}_{\mathcal{D}^+(\mathcal{C}_0)}(X, I) \simeq \operatorname{Hom}_{\mathcal{C}}(X, I)$ as spectra. Thus, the vanishing condition follows from the assumption that F is fully faithful. For the converse, let us suppose that the vanishing condition holds; we will show that the natural map of spectra $\operatorname{Hom}_{\mathcal{D}^+(\mathcal{C}_0)}(X, Y) \simeq \operatorname{Hom}_{\mathcal{C}}(X, Y)$ is an equivalence for all $X, Y \in \mathcal{D}^+(\mathcal{C}_0)$.

We note that, by the construction of F, we have natural isomorphisms $\pi_i FX \simeq \pi_i X$ in $\mathcal{C}_0$. Since both $\mathcal{D}^+(\mathcal{C}_0)$ and $\mathcal{C}^+$ are right-complete, we see that $X = \operatorname{colim}\{\tau_{\geq -n} X\}$ and

$$FX = \operatorname{colim}\{\tau_{\geq -n} FX\} = \operatorname{colim}\{F \tau_{\geq -n} X\}.$$

Thus we may reduce to the case where X is bounded. Working by induction, we may reduce to the case where X is concentrated in a single degree. Shifting, we may suppose that $X \in \mathcal{C}_0$.

Replacing Y by $Y[-n]$ for $n \gg 0$, we may suppose that Y is representable by an injective complex concentrated in (homological) degrees ≤ 0. Let $I^\bullet$ denote the associated cosimplicial objects of $\mathcal{C}_0^{\text{inj}}$. Then Y is the geometric realization of the cosimplicial object $I^\bullet$, and by construction FY is the geometric realization of $FI^\bullet$. We may therefore reduce to the case where $Y \in \mathcal{C}_0^{\text{inj}}$. But we have already noted that the case $X \in \mathcal{C}_0$, $Y \in \mathcal{C}_0^{\text{inj}}$ is equivalent to the vanishing hypothesis given in the theorem.

To complete the proof, we show that if F is fully faithful then it is also essentially surjective. Let $X \in \mathcal{C}^+$; we wish to show that X belongs to the essential image of F. First suppose that $X \in \mathcal{C}^b$. We work by induction on the number indices i for which $\pi_i X \neq 0$. If X has only one nonvanishing homotopy group, then X belongs to some shift of $\mathcal{C}_0$ and obviously lies in the essential image of F. Otherwise, we may suppose that there exists a triangle $X' \to X \to X''$, where X' and X'' belong to the essential image of F. Now X may be described as the kernel of a map $X'' \to X'[1]$. Since F is fully faithful, the map $X'' \to X'[1]$ is obtained by applying F to some map $Y'' \to Y'[1]$. Then the kernel of this map is preimage for X under F.

In the general case, we may write X as the colimit of the sequence $\{\tau_{\geq -n} X\}$. Each term of this sequence lies in $\mathcal{C}^b$, so we may write $\tau_{\geq -n} X \simeq FY_n$. Let Y be the colimit of the sequence Y_n in $\mathcal{D}(\mathcal{C}_0)$. Then by checking on homotopy groups, we see that $Y \in \mathcal{D}^+$ and the natural map $X \to FY$ is an equivalence. □

Remark 2.3.7. For $X, I \in \mathcal{C}_0$, the group $\pi_0 \operatorname{Hom}_{\mathcal{C}}(X, I[n])$ is closely related to the Yoneda-Ext group $\operatorname{Ext}^n_{\mathcal{C}_0}(X, I)$. If $\mathcal{C}$ is the derived category of its heart, then these two groups coincide. Proposition 2.3.6 asserts that the converse holds (at least for left-bounded objects) provided that $\pi_0 \operatorname{Hom}_{\mathcal{C}}(X, I[n])$ has one property in common with $\operatorname{Ext}^n_{\mathcal{C}_0}(X, I)$: it must vanish when I is injective and $n > 0$.

Remark 2.3.8. If the conditions of the last clause of Proposition 2.3.6 are satisfied, then $\mathcal{C}$ is equivalent to $\mathcal{D}(\mathcal{C}_0)$ if and only if $\mathcal{C}$ is left complete.

2.4 A_∞-Ring Spectra

Let $\mathcal{C}$ be a stable ∞-category, and let $X \in \mathcal{C}$ be an object. Since $\mathcal{C}$ is naturally enriched over spectra, we can extract a spectrum $\operatorname{End}_{\mathcal{C}}(X)$ whose 0th space is given by $\operatorname{Hom}_{\mathcal{C}}(X, X)$. By analogy with the theory of ordinary abelian categories, we would expect that $\operatorname{End}_{\mathcal{C}}(X)$ has the structure of an associative ring, in some reasonable "up to homotopy" sense. The correct way of describing the situation is to say that $\operatorname{End}_{\mathcal{C}}(X)$ is an A_∞-ring spectrum.

We shall not give the precise definition of an A_∞-ring spectrum here. Let us simply remark that it is not sufficient to consider a "monoid object" R in the homotopy category of spectra with respect to the smash product (which we shall denote by $\otimes$ to be consistent with our earlier notation). This gives rise to the notion of a *homotopy associative* ring spectrum which is too crude for sophisticated algebraic purposes. Homotopy associativity is the assumption that the two natural maps $R \otimes R \otimes R \to R$ (given by iterated composition) are homotopic to one another. For most applications one needs also to know the homotopy, and to know that this homotopy satisfies certain higher associativity conditions of its own.

The appropriate associativity conditions were first formulated by Stasheff (see [34]) in terms of certain higher-dimensional polytopes which are now called *Stasheff associahedra*. A simpler formulation was later given in terms of operads, leading to the definition of an A_∞-ring spectrum as an algebra over an appropriate A_∞-operad. This point of view has the merit that it can be used to describe other, more subtle types of algebraic structure (such as the notion of an E_n-ring spectrum obtained from the "little n-cubes operad", which governs the structure of n-fold loop spaces).

In recent years, it has become possible to give an even simpler approach to the theory of A_∞ ring spectra, based on new models for the stable homotopy category which are equipped with a smash product functor which is associative (and commutative) *on the nose* (see, for example, [9]). In one of these settings, one can speak of spectra equipped with multiplications which are strictly associative, and this turns out to be equivalent to requiring associativity up to all higher homotopies.

We will be content to simply describe A_∞-ring spectra and how to work with them. The intuition is that they behave like a somewhat sophisticated version of associative rings.

The first thing to be aware of is that an A_∞-ring spectrum A has an underlying spectrum. This spectrum has homotopy groups $\{\pi_i A\}_{i \in \mathbf{Z}}$, all of which are abelian. Moreover, the "ring structure" on A induces a ring structure (in the ordinary sense) on $\oplus_{i \in \mathbf{Z}} \pi_i A$, which is compatible with the $\mathbf{Z}$-grading. In particular, $\pi_0 A$ is an ordinary associative ring and each $\pi_i A$ has the structure of a bimodule over $\pi_0 A$.

An A_∞-ring spectrum is said to be *connective* if its underlying spectrum is connective: that is, if $\pi_i A = 0$ for $i < 0$. Just as a connective spectrum can be thought of as a space equipped with an addition which is commutative and associative up to homotopies of all orders, a connective A_∞-ring spectrum can be thought of as a space equipped with an addition and multiplication which are commutative (for the addition only) and associative (for both the addition and the multiplication) up to all higher homotopies.

Most of the A_∞-ring spectra which we shall meet will be algebras over the ordinary commutative ring $\mathbf{Z}$. This implies that there exists a map of A_∞-ring spectra $f : \mathbf{Z} \to A$ (which is not automatic: for example, such a map does not exist when A is the sphere spectrum). However, f does not determine a $\mathbf{Z}$-algebra structure on A; one also needs to know that f is *central* in some sense. In the context of structured ring spectra this is not simply a *condition* on f, but consists of extra data which must be supplied.

Connective A_∞-$\mathbf{Z}$-algebras, it turns out, are easy to think about. To begin with, the underlying spectrum of a $\mathbf{Z}$-algebra A is not arbitrary, but must be equipped with the structure of a module over $\mathbf{Z}$. This forces the underlying spectrum of A to be a "generalized Eilenberg-MacLane spectrum" (that is, equivalent to a product of Eilenberg-MacLane spectra). In the case where A is connective, if we think of the underlying spectrum as a space

X with a coherently commutative addition law, then giving a $\mathbf{Z}$-module structure on A is essentially equivalent to giving a model X where the addition is commutative (and associative) on the nose. In the case where A is a connective A_∞-ring spectrum, a $\mathbf{Z}$-algebra structure on A allows us to model A using a topological space which is equipped with an addition and multiplication that are both associative and commutative (for the addition) *on the nose.* In other words, a connective A_∞-$\mathbf{Z}$-algebra is more or less the same thing as an associative topological ring. If we use simplicial sets in place of topological spaces as models for homotopy theory, then we obtain an analogous result: a connective A_∞-$\mathbf{Z}$-algebra is more or less the same thing as a simplicial (associative) ring.

For any A_∞-ring spectrum A, there exists a good theory of *left A-module spectra*, or simply *left A-modules.* The collection of left A-modules forms a presentable, stable ∞-category $\mathcal{M}_A$. If M is an A-module, then M has an underlying spectrum and we will write $\pi_i M$ for the homotopy groups of this underlying spectrum. We call an A-module M *n-connected* if $\pi_i M = 0$ for $i \leq n$, and *n-truncated* if $\pi_i M = 0$ for $i > n$. A morphism $f : M \to N$ of left A-modules is said to be *n-connected* if its cokernel is n-connected. If A is connective, then the collections $\{M : (\forall i > 0)[\pi_i M = 0]\}$ and $\{M : (\forall i < 0)[\pi_i M = 0]\}$ determine an admissible t-structure on $\mathcal{M}_A$. With respect to this t-structure, $\mathcal{M}_A$ is left-complete and right-complete, and the formation of homotopy groups is compatible with the formation of filtered colimits. The heart of $\mathcal{M}_A$ is equivalent to the abelian category of discrete left modules over the ring $\pi_0 A$. For any left A-module M, the direct sum $\oplus_i \pi_i M$ forms a graded left module over the graded ring $\oplus_i \pi_i A$. We say that M is *connective* if $\pi_i M = 0$ for $i < 0$, and *discrete* if $\pi_i M = 0$ for $i \neq 0$. In the case where A is a connective $\mathbf{Z}$-algebra, we can model A by a topological ring, and one can think of connective A-modules as topological modules over this topological ring. When A is a discrete ring, the ∞-category of A-module spectra is the derived category of the (Grothendieck) abelian category of discrete A-modules. If K is a complex of (ordinary) A-modules, thought of as an A-module spectrum, then its homotopy groups are given by $\pi_i K = \mathrm{H}^{-i}(K)$.

The ∞-category of left A-modules has enough compact objects, which are called *perfect* A-modules. The class of perfect A-modules form a stable subcategory $\mathcal{M}_A^{pf} \subseteq \mathcal{M}_A$ containing A, and $\mathcal{M}_A \simeq \mathrm{Ind}(\mathcal{M}_A^{pf})$. We call an A-module *finitely presented* if it lies in the smallest stable subcategory of $\mathcal{M}_A$ containing A. An A-module is *perfect* if and only if it is a retract of a finitely presented A-module.

Example 2.4.1. If A is a discrete associative ring, then an A-module is finitely presented if it can be represented by a finite complex of finitely generated free A-modules, and perfect if it can be represented by a finite complex of finitely generated projective A-modules.

Let $\mathcal{C}$ be any stable ∞-category, and let $X \in \mathcal{C}$. As suggested above, the spectrum $\mathrm{End}_{\mathcal{C}}(X)$ has an A_∞-ring structure. The full subcategory of $\mathcal{C}$ consisting of the object X is equivalent to the full subcategory of $\mathcal{M}_A$ consisting of the trivial A-module A. This equivalence extends to an exact, fully faithful functor F (a kind of "external tensor product by X") from the ∞-category of finitely presented A-modules to $\mathcal{C}$. If $\mathcal{C}$ is stable under the formation of retracts, then F extends uniquely to $\mathcal{M}_A^{pf}$ (and remains exact and fully faithful).

If $\mathcal{C}$ is stable under the formation of sums, then F extends to an exact functor on $\mathcal{M}_A$, which is fully faithful provided that X is a compact object of $\mathcal{C}$.

There is a theory of *right A-module spectra* which is entirely dual to the above theory of left A-modules spectra; it may also be regarded as the theory of left modules over an opposite A_∞-ring spectrum A^{op}. As with ordinary algebra, if M is a right module over A and N is a left module over A, then one can define the tensor product $M \otimes_A N$. In general, $M \otimes_A N$ is merely a spectrum with no A-module structure. The functor $(M, N) \mapsto M \otimes_A N$ is exact and colimit-preserving in both variables. There is a spectral sequence for computing the homotopy groups of $M \otimes_A N$, with E_2-term given by $E_2^{p\bullet} = \mathrm{Tor}_p^{A_\bullet}(\pi_\bullet M, \pi_\bullet N)$. Here the notation is intended to indicate that one computes the Tor_p-group in the context of graded modules over a graded ring, and consequently it comes equipped with a natural grading: E_2^{pq} is the qth graded piece. This spectral sequence is strongly convergent provided that A, M, and N are connective. In particular, if A, M, and N are connective, then the spectrum $M \otimes_A N$ is connective, and $\pi_0(M \otimes_A N)$ is naturally isomorphic to the ordinary tensor product $\mathrm{Tor}_0^{\pi_0 A}(\pi_0 M, \pi_0 N)$. If A is discrete, then $\otimes_A$ is the left-derived functor (in either variable) of the ordinary tensor product, so that if M and N are discrete also we have $\pi_i(M \otimes_A N) \simeq \mathrm{Tor}_i^A(M, N)$.

If $A \to B$ is a morphism of A_∞-ring spectra, then we may regard any B-module as an A-module by restriction of structure. This restriction functor has both a left adjoint and a right adjoint (and is therefore exact), which we shall denote by $M \mapsto B \otimes_A M$ and $M \mapsto \mathrm{Hom}_A(B, M)$.

Remark 2.4.2. The standard notation in homotopy theory is to write $B \wedge_A M$, rather than $B \otimes_A M$. We shall instead employ the usual algebraic notation, which we feel is easier to read and better brings out the analogy with the classical algebraic notion of tensor product. However, we warn the reader to keep in mind that our tensor products are not the usual tensor products of algebra but suitable left-derived analogues.

2.5 Properties of A_∞ Ring Spectra and their Modules

Let A be a connective A_∞-ring spectrum. We are going to discuss some basic facts about the stable ∞-category of left A-modules. We call an A-module *free* if it is a direct sum of (unshifted) copies of A. We call a map $N \to N'$ of connective A-modules *surjective* if it induces a surjection on π_0.

Proposition 2.5.1. *Let M be a connective left A-module. Then the following are equivalent:*

- *The module M is a retract of a free A-module $\bigoplus_{i \in I} A$.*
- *For any surjection $f : N \to N'$ of connective left A-modules, the induced map $\pi_0 \mathrm{Hom}_A(M, N) \to \pi_0 \mathrm{Hom}_A(M, N')$ is surjective.*

Proof. It is clear that the first condition implies the second. For the converse, choose a surjection $N \to M$ with N free, and apply the surjectivity assumption to the identity in $\mathrm{Hom}(M, M)$. □

We shall call a connective left A-module *projective* if it satisfies the above conditions.

Theorem 2.5.2 (Derived Lazard Theorem). *Let R be a connective A_∞-ring spectrum, and let M be a connective left R-module. The following conditions are equivalent:*

1. *The module M is a filtered colimit of finitely generated free modules.*
2. *The module M is a filtered colimit of projective modules.*
3. *If N is a discrete right R-module, then $N \otimes_R M$ is discrete.*
4. *The $\pi_0 R$-module $\pi_0 M$ is flat, and the natural map $\mathrm{Tor}_0^{\pi_0 R}(\pi_i R, \pi_0 M) \to \pi_i M$ is an isomorphism for each $i \geq 0$.*
5. *The $\pi_0 R$-module $\pi_0 R \otimes_R M$ is discrete and flat (in the sense of ordinary commutative algebra).*

Proof. It is obvious that (1) implies (2).

If M is a free left R-module, then for any right R-module N, $N \otimes_R M$ is a direct sum of copies of N, hence is discrete provided that N is discrete. Since the formation of tensor products is compatible with filtered colimits and filtered colimits of discrete R-modules are discrete, we deduce that (2) implies (3).

Suppose that (3) is satisfied. We may identify discrete right R-modules with discrete right $\pi_0 R$-modules. If N is discrete, then the discrete module $N \otimes_R M$ is equivalent to $\mathrm{Tor}_0^{\pi_0 R}(N, \pi_0 M)$. It follows that the functor $\mathrm{Tor}_0^{\pi_0 R}(\bullet, \pi_0 M)$ is an exact functor so that $\pi_0 M$ is flat over $\pi_0 R$. Now one can prove by induction on i that for any connective right R-module N, the natural maps $\mathrm{Tor}_0^{\pi_0 R}(\pi_i N, \pi_0 M) \to \pi_i(N \otimes_R M)$ are isomorphisms. Applying this in the case $N = R$, we deduce (4).

Any discrete right R-module may be considered as a $\pi_0 R$-module, and we have $\pi_i(N \otimes_R M) = \pi_i(N \otimes_{\pi_0 R} \pi_0 R \otimes_R M) = \mathrm{Tor}_i^{\pi_0 R}(N, \pi_0 R \otimes_R M)$. Thus (3) is equivalent to (5).

To complete the proof, it will suffice to show that (4) implies (1). Let $\mathcal{C}$ denote the ∞-category of finitely generated free left R-modules equipped with maps to M. Then $\mathcal{C}$ is essentially small, and gives rise to a diagram in the category of left R-modules which has a colimit M'. By construction there is a natural map $\phi : M' \to M$. We will complete the proof by showing that if the (4) is satisfied, then $\mathcal{C}$ is filtered and ϕ is an equivalence.

It is clear that any two objects of $\mathcal{C}$ admit a map to a common third object of $\mathcal{C}$ (use the direct sum). To complete the proof, it suffices to show that for any $K, K' \in \mathcal{C}$ and any element $\eta : S^n \to \mathrm{Hom}_{\mathcal{C}}(K, K')$, there exists a morphism $K' \to K'''$ in $\mathcal{C}$ such that the image of η in $\pi_n \mathrm{Hom}(K, K''')$ vanishes. For $n = 0$, this follows from the classical version of Lazard's theorem (see [21]); hence we shall assume $n > 0$. The module K is a direct sum of finitely many copies of R. Arguing iteratively, we can reduce to the case $K = R$. In this case, $\mathrm{Hom}_{\mathcal{C}}(K, K')$ may be identified with the homotopy fiber of the structural map $K' \to M$. The image of η in $\pi_n K'$ lies in the kernel of the natural map $\pi_n K' \to \pi_n M = \mathrm{Tor}_0^{\pi_0 R}(\pi_n R, \pi_0 M)$. Since $\pi_0 M$ is a filtered colimit of free $\pi_0 R$ modules, it follows by a direct limit argument that there exists a free $\pi_0 R$-module K_0 and a factorization $\pi_0 K' \to K_0 \to M$ such that the image of η in $\text{Tor}_0^{\pi_0 R}(\pi_n R, \pi_0 K')$ vanishes in $\text{Tor}_0^{\pi_0 R}(\pi_n R, \pi_0 K_0)$. Using the freeness of K' and K_0, we can lift K_0 to a free R-module K'' and obtain a factorization $K' \to K'' \to M$, such that the image of η in $\pi_n K''$ vanishes.

Now the exact sequence $\pi_{n+1} K'' \to \pi_{n+1} M \to \pi_n \operatorname{Hom}_{\mathcal{C}}(K, K'') \to \pi_n K''$ shows that η is the image of a class $\widetilde{\eta} \in \pi_{n+1} M = \pi_{n+1} R \otimes_{\pi_0 R} \pi_0 M$. Applying a direct limit argument again, we can find a factorization $K'' \to K''' \to M$ such that $\widetilde{\eta}$ is in the image of $\pi_{n+1} K''' \to M$. It then follows that the image of η in $\pi_n \operatorname{Hom}_{\mathcal{C}}(K, K''')$ vanishes, as desired.

It follows that $\mathcal{C}$ is filtered. To complete the proof, we will show that ϕ is an equivalence. Since we have $\text{Tor}_0^{\pi_0 R}(\pi_n R, \pi_0 M) \simeq \pi_n M$ and $\text{Tor}_0^{\pi_0 R}(\pi_n R, \pi_0 M') \simeq \pi_n M'$, it will suffice to show that ϕ induces an isomorphism $\pi_0 \phi : \pi_0 M' \to \pi_0 M$. It is obvious that $\pi_0 \phi$ is surjective (consider free modules of rank one). To prove injectivity, we represent any given element of $\pi_0 M'$ by $\zeta \in \pi_0 K$ for some finitely generated free left R-module K equipped with a map to M. Now if the image of ζ in $\pi_0 M$ vanishes, then by Lazard's theorem there is a finitely generated free left $\pi_0 R$-module K'_0 and a factorization $\pi_0 K \to K'_0 \to \pi_0 M$ such that ζ vanishes in K'_0. Using the freeness of K and K'_0, we lift the factorization to a diagram $K \to K' \to M$. Thus the natural map $\pi_0 K \to \pi_0 M'$ factors through $\pi_0 K'$ and therefore kills ζ. □

A connective left R-module satisfying the above hypotheses will be said to be *flat*. In general, if M is a flat left R-module, then the "global" properties of M as an R-module are determined by the "local" properties of $\pi_0 M$ as a module over the ordinary ring $\pi_0 R$. As an illustration of this principle, we note that:

Proposition 2.5.3. *Let R be a connective A_∞-ring spectrum. A flat left R-module M is projective if and only if $\pi_0 M$ is a projective $\pi_0 R$ module.*

Proof. We first suppose that $\pi_0 M$ is free. In this case, we may choose generators for $\pi_0 M$ over $\pi_0 A$, and lift these to obtain a map $f : \oplus_i A \to M$ which induces an isomorphism on π_0. Since both the source and target of f are flat, it follows that f is an equivalence so that M is free.

In the general case, we choose a surjection from a free module onto $\pi_0 M$ whose kernel is also free (this can be achieved using an "Eilenberg-swindle" argument), and lift this to a surjection $f : F \to M$. Since M is flat and f induces a surjection on homotopy groups in each degree, we deduce that $F' = \ker f$ is flat and that

$$0 \to \pi_0 F' \to \pi_0 F \to \pi_0 M \to 0$$

is exact. Since $\pi_0 M$ is projective, the inclusion $\pi_0 F' \subseteq \pi_0 F$ is split by a morphism which lifts to a map $\phi : F \to F'$. Let ψ denote the natural map $F' \to F$. Then $\phi \circ \psi$ induces the identity on $\pi_0 F'$, and therefore it is homotopic to the identity on F' since F' is free. It follows that the triangle $F' \to F \to M$ is split, so that M is retract of F. Since F is free, M is projective. □

Proposition 2.5.4. *Let A be a connective A_∞-ring spectrum and let M be a flat left A-module. The following conditions are equivalent:*

- *For any nonzero right A-module N, the spectrum $N \otimes_A M$ is nonzero.*

- *The $\pi_0 A$ module $\pi_0 M$ is faithfully flat over A.*

Under these circumstances, we shall say that M is *faithfully flat* over A. Let $f : A \to B$ be a morphism of A_∞-ring spectra. We say that B is (faithfully) flat if it is (faithfully) flat as an A-module (we really have two notions, depending on whether we choose to view B as a left or as a right A-module: unless otherwise specified, we regard B as a *left* A-module).

We now discuss some finiteness conditions on left A-modules. We have already remarked that $\mathcal{M}_A$ has enough compact objects, and that these are called *perfect* A-modules. We will characterize the perfect A-modules as those which admit preduals. We first mention the following general principle:

Proposition 2.5.5. *Let A be an A_∞-ring spectrum. The following ∞-categories are naturally equivalent:*

1. *The ∞-category of left exact functors $\mathcal{M}_A \to \mathcal{S}$ which commute with filtered colimits.*

2. *The ∞-category of exact functors from $\mathcal{M}_A$ into the ∞-category $\mathcal{S}_\infty$ of spectra which commute with filtered colimits.*

3. *The ∞-category of right A-modules.*

Proof. Let $\mathcal{C}_{(1)}$, $\mathcal{C}_{(2)}$, and $\mathcal{C}_{(3)}$ be the three ∞-categories described in the statement of the proposition. We give a sketch of the construction of functors which give rise to the equivalences $\mathcal{C}_{(1)} \simeq \mathcal{C}_{(2)} \simeq \mathcal{C}_{(3)}$.

First of all, composition with the functor $\Omega^\infty : \mathcal{S}_\infty \to \mathcal{S}$ (passage to the zeroth space) gives us a "forgetful functor" $\phi : \mathcal{C}_{(2)} \to \mathcal{C}_{(1)}$. The crucial observation is that since $\mathcal{M}_A$ is stable, ϕ is not as forgetful as it looks. In fact, any left-exact functor $F : \mathcal{M}_A \to \mathcal{S}$ may be uniquely "enriched" to an exact functor $F_\infty : \mathcal{M}_A \to \mathcal{S}_\infty$ with $F \simeq \Omega^\infty \circ F_\infty$. We first note that $\mathcal{M}_A$ has a zero object. Since F is left exact, $F0$ is a final object $* \in \mathcal{S}$. The unique maps $0 \to M$ give rise to maps $* \simeq F0 \to FM$ for each $M \in \mathcal{M}_A$, so that F admits a natural factorization $\widetilde{F} : \mathcal{M}_A \to \mathcal{S}_*$ through the ∞-category of *pointed* spaces.

We now note that since M is the n-fold loop space of $M[n]$ in $\mathcal{M}_A$, $\widetilde{F}M \simeq \Omega^n \widetilde{F}M[n]$. In other words, the functor $\widetilde{F}M$ comes equipped with a sequence functorial deloopings of the spaces $\widetilde{F}M$. We may now define the functor F_∞ by letting $F_\infty M$ be the spectrum corresponding to this sequence of deloopings $(\widetilde{F}M, \widetilde{F}M[1], \ldots)$. Since F is left exact, $\widetilde{F}$ is exact. Moreover, $\widetilde{F}$ will commute with filtered colimits if and only if F commutes with filtered colimits. This proves the equivalence of $\mathcal{C}_{(1)}$ and $\mathcal{C}_{(2)}$ (not that so far, we have used only the stability of $\mathcal{M}_A$).

We note that there is a natural functor $\mathcal{C}_{(3)} \to \mathcal{C}_{(2)}$, which carries a right A-module N to the functor $M \mapsto N \otimes_A M$. The inverse functor $\mathcal{C}_{(2)} \to \mathcal{C}_{(3)}$ is given by evaluation at the identity object A. More specifically, we note that A is a right A-module in the ∞-category of left A-modules. Consequently, if $F \in \mathcal{C}_{(2)}$, then $F(A)$ is a right A-module spectrum. Moreover, one can construct a natural transformation $F(A) \otimes_A M \to F(M)$. This transformation is an equivalence when $M = A$. Since both sides are exact as a functor of M, we deduce that this map is an equivalence whenever M is finitely presented. Finally, if F commutes with filtered colimits, then both sides are an equivalence in general. □

Corollary 2.5.6. *Let A be an A_∞-ring spectrum, and let M be a left A-module. Then M is perfect if and only if there exists a right A-module M^* and an identification of spectrum-valued functors $\mathrm{Hom}_{\mathcal{M}_A}(M, \bullet) \simeq M^* \otimes \bullet$.*

Proof. Apply Proposition 2.5.5 to the functor $\mathrm{Hom}_{\mathcal{M}_A}(M, \bullet)$. □

In particular, applying both sides to the left A-module A, we see that $M^* = \mathrm{Hom}(M, A)$ is the dual of M in the usual sense.

We next discuss a somewhat weaker finiteness condition.

Proposition 2.5.7. *Let A be a connective A_∞-ring spectrum, let M be a left A-module, and let n be an integer. The following conditions are equivalent:*

1. *There exists a finitely presented left A-module N and a morphism $N \to M$ which induces an equivalence $\tau_{\leq n} N \simeq \tau_{\leq n} M$.*
2. *There exists a perfect left A-module N and a morphism $N \to M$ which induces an equivalence $\tau_{\leq n} N \simeq \tau_{\leq n} M$.*
3. *There exists a finitely presented left A-module N and an equivalence $\tau_{\leq n} N \simeq \tau_{\leq n} M$.*
4. *There exists a perfect A-module N and an equivalence $\tau_{\leq n} N \simeq \tau_{\leq n} M$.*
5. *The n-truncated A-module $\tau_{\leq n} M$ is a compact object of $(\mathcal{M}_A)_{\leq n}$.*

Proof. It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). Since $(\mathcal{M}_A)_{\leq n}$ is stable under filtered colimits, the implication (4) $\Rightarrow$ (5) follows from the fact that $\mathrm{Hom}_{\mathcal{M}_A}(M, M') = \mathrm{Hom}_{\mathcal{M}_A}(\tau_{\leq n} M, M')$ when M' is n-truncated.

Suppose that (5) is satisfied by M. We first claim that M is k-connected for sufficiently small k. Since $\tau_{\leq n} M$ is the filtered colimit of the A-modules $\tau_{\geq k}(\tau_{\leq n} M)$ as $k \to \infty$, it follows from (5) that the identity map from $\tau_{\leq n} M$ to itself factors through $\tau_{\geq k}(\tau_{\leq n} M)$ for some k. Thus the identity map on $\pi_m M$ factors through 0 for $m < k$, which implies that M is $(k-1)$-connected.

We now prove that for all $m \leq n$, there exists a finitely presented left A-module N_m and a map $\phi_m : N_m \to M$ which induces an isomorphism on homotopy groups in degrees $\leq m$. If m is sufficiently small, then M is m-connected and we may take $N_m = 0$. The proof in general goes by induction on m. Suppose that we have constructed $\phi_{m-1} : N_{m-1} \to M$, and let K denote the cokernel of ϕ_{m-1}. Then K also has the property stated in (5). Since $m \leq n$, we deduce that $\pi_m K$ is a compact object in the ordinary category of discrete $\pi_0 A$-modules. Thus there exists a presentation

$$P_0 \to Q_0 \to \pi_m K \to 0$$

for $\pi_m K$, where P_0 and Q_0 are free $\pi_0 A$-modules. We may lift the generators of P_0 and Q_0 to obtain a triangle (not necessarily exact) of left A-modules

$$P[m] \xrightarrow{\psi} Q[m] \to K$$

Since $N_{m-1} \to M$ induces an isomorphism on π_{m-1}, the long exact sequence implies that $\pi_m M \to \pi_m K$ is surjective. Since Q is free, $Q[m] \to K$ factors through some map $\theta : Q[m] \to M$. Since the composition $P[m] \to Q[m] \to M \to K$ is zero, $P[m] \to M$ factors through some map $\psi' : P[m] \to N_{m-1}$. We let N_m denote the cokernel of

$$P[m] \xrightarrow{\psi \oplus \psi'} Q[m] \oplus N_{m-1}.$$

By construction, composition with the map $(-\theta) \oplus \phi_{m-1} : Q[m] \oplus N_{m-1} \to M$ kills $\psi \oplus \psi'$, and therefore $(-\theta) \oplus \phi_{m-1}$ factors through N_m. A simple diagram chase shows that any factorization $N_m \to M$ induces an isomorphism on homotopy groups in dimension $\leq m$. This completes the induction, and taking $m = n$ completes the proof of (5) $\Rightarrow$ (1). □

We will say that an A-module M is *perfect to order n* if it satisfies the equivalent conditions of Proposition 2.5.7. We shall say that M is *almost perfect* if it is perfect to order n for all n. We note that by definition, the class of modules which are perfect to order n is stable under finite colimits. Thus the class of almost perfect modules is stable under finite colimits and shifts, and therefore constitutes a stable subcategory of $\mathcal{M}_A$.

Remark 2.5.8. We have now discussed a great number of finiteness conditions on A-modules, and it seems worthwhile to discuss the relationships between them. An A-module is finitely presented if it can be built using only finitely many cells, and perfect if it is a retract of a finitely presented module. An A-module is almost perfect if it admits a cell decomposition in which the dimensions of the cells tend to ∞: in other words, we allow infinitely many cells, but only finitely many cells of dimension $\leq n$ for any fixed n. Finally, we say that an A-module is perfect to order n if it can be built using only finitely many cells of dimension $\leq n+1$ (but possibly more cells of larger dimension).

The notion of perfect to order n will be needed for certain approximation arguments later in this paper, to eliminate Noetherian hypotheses. The reader who is not interested in these applications should feel free to ignore this slightly technical notion.

We now discuss Noetherian conditions on A_∞-ring spectra. Recall that an ordinary ring R is said to be *left coherent* if every finitely generated left ideal of R is finitely presented.

Definition 2.5.9. A connective A_∞-ring spectrum A is *left Noetherian* (*left coherent*) if $\pi_0 A$ is left Noetherian (left coherent), and each $\pi_n A$ is a finitely presented left $\pi_0 A$-module. If A is coherent and M is a left A-module, then we say that M is *coherent* if each $\pi_n M$ is a finitely presented as a discrete module over $\pi_0 A$.

We note that if A is left coherent, then the coherent left A-modules form a stable subcategory of $\mathcal{M}_A$ which is stable under the formation of retracts. This subcategory includes A, and therefore includes all perfect A-modules.

Proposition 2.5.10. *Suppose that A is left coherent. Let M be an A-module. Then M is almost perfect (perfect to order n) if and only if the following conditions are satisfied:*

- *For $m \ll 0$, $\pi_m M = 0$.*
- *For all $m \in \mathbf{Z}$ ($m \leq n$), $\pi_m M$ is a finitely presented $\pi_0 A$-module.*

Proof. Suppose that M is perfect to order n. Then there exists a perfect A-module N and an equivalence $\tau_{\leq n} M \simeq \tau_{\leq n} N$. Replacing M by N, we may suppose that M is perfect. Then the first condition is clear and the second condition holds since M is coherent.

Now suppose that $\pi_m M$ is finitely presented over $\pi_0 A$ for $m \leq n$, and vanishes for $m \ll 0$. The proof that M is perfect to order n is identical to the proof of the implication (5) $\Rightarrow$ (1) of Proposition 2.5.7. □

Proposition 2.5.11. *Let A be a connective A_∞ ring spectrum. The following are equivalent:*

1. *A is left coherent.*
2. *For any almost perfect $M \in \mathcal{M}_A$, the truncation $\tau_{\geq 0} M$ is almost perfect.*

Proof. The implication (1) $\Rightarrow$ (2) follows from the description of almost perfect modules given in Proposition 2.5.10. Conversely, suppose that (2) is satisfied. We note that the first non-vanishing homotopy group of any almost perfect A-module is a finitely presented module over $\pi_0 A$ in the usual sense. Applying (2) to the module $A[-n]$, we deduce that $\pi_n A$ is a finitely presented $\pi_0 A$-module. To complete the proof, it suffices to show that $\pi_0 A$ is left coherent.

Clearly, (2) implies that $\pi_0 A$ is almost perfect as an A-module. From this we may deduce that any almost perfect $\pi_0 A$-module is almost perfect as an A-module. Thus, we may replace A by $\pi_0 A$ and reduce to the case where A is discrete.

Let $I \subseteq A$ be a finitely generated ideal. Then I is the image of some map $\phi : A^n \to A$; we wish to show that the kernel of ϕ is finitely generated. But the kernel of ϕ is $\tau_{\geq 0} K$, where K is the kernel of ϕ in $\mathcal{M}_A$. Since K is perfect, condition (2) implies that $\tau_{\geq 0} K$ is almost perfect, hence $\pi_0 \tau_{\geq 0} K$ is finitely presented. □

If A is left coherent and $M \in \mathcal{M}_A$, then we shall say that M is *coherent* if $\pi_n M$ is a finitely presented module over $\pi_0 A$ for each n. Equivalently, M is coherent if $\tau_{\geq n} M$ is almost perfect for each $n \in \mathbf{Z}$. The property of coherence is not stable under base change. However, it is stable under a (right)-flat base change $A \to B$ of coherent (connective) A_∞-ring spectra: this follows immediately from the characterization given in Proposition 2.5.10.

Proposition 2.5.12. *Let A be a connective A_∞-ring spectrum. Let M be a flat left A-module which is almost perfect. Then M is perfect and projective.*

Proof. We note that $\pi_0 M$ is a finitely presented, flat $\pi_0 A$-module. Consequently, $\pi_0 M$ is projective, so that M is projective. We may choose a surjection from a finitely generated free module F onto M; the projectivity of M implies that M is a retract of F. Since F is perfect, so is M. □

In the future, we will need a slight generalization of Proposition 2.5.12. We will say that a left A-module M has Tor-*amplitude* $\leq n$ if, for any discrete right A-module N, the groups $\pi_i(N \otimes_A M)$ vanish for $i > n$.

Proposition 2.5.13. *Let A be a connective A_∞-ring spectrum.*

- *If M is an A-module of* Tor-*amplitude* $\leq n$, *then* $M[k]$ *has* Tor-*amplitude* $\leq n + k$.
- *Let $M' \to M \to M''$ be an exact triangle of A-modules. If M' and M'' have* Tor-*amplitude* $\leq n$, *then so does M.*
- *If M is almost perfect and has* Tor-*amplitude* $\leq n$, *then M is perfect.*

Proof. The first two claims are obvious. For the third, we first shift M so that it is connective, and then work by induction on the Tor-amplitude of M. Choosing a surjection $F \to M$, we note that if M has Tor-amplitude $\leq n$ and $n > 0$, then the kernel K of the surjection is connective and of Tor-amplitude $\leq (n-1)$. In this fashion we reduce to the case where M is connective of Tor-amplitude ≤ 0. Then M is flat and we may apply Proposition 2.5.12. □

2.6 E_∞-Ring Spectra and Simplicial Commutative Rings

Just as commutative algebra provides the foundation for classical algebraic geometry, our theory of derived algebraic geometry will require some kind of "derived commutative algebra", in which commutative rings are replaced by an appropriate homotopy-theoretic generalization. However, this turns out not to such a simple story, since there are several plausible candidates for this generalization. The objective of this section is to explain what these candidates are and how they are related to one another, and to explain why we believe that one formalism (that of simplicial commutative rings) provides the proper foundation for the theory that we will develop later.

In the last two sections, we have discussed A_∞-ring spectra, which are a good homotopy-theoretic generalization of associative rings. However, in algebraic geometry we need to deal with *commutative* rings, and their homotopy-theoretic generalizations are considerably more subtle. Fix an ordinary commutative ring R. Then there exist (at least) three homotopy-theoretic generalizations of the notion of "commutative R-algebra":

- One can consider topological commutative rings endowed with an R-algebra structure. These form an ∞-category which we shall denote by $\mathcal{SCR}_{R/}$. Objects of $\mathcal{SCR}_{R/}$ can also be modelled by *simplicial commutative R-algebras.* The category of simplicial commutative R-algebras has a Quillen model structure in which the weak equivalences and the fibrations are those maps which are weak equivalences or fibrations on the underlying simplicial sets. We will discuss this ∞-category at great length in the next section (and throughout the remainder of this paper).

- One can consider commutative differential graded R-algebras. More precisely, one can consider a simplicial localization of the ordinary category of differential graded R-algebras which inverts quasi-isomorphisms. There exists a Quillen model structure on the category of differential graded R-algebras, with quasi-isomorphisms as weak equivalences and cofibrations given by retracts of iterated cell attachments. If R is a $\mathbf{Q}$-algebra, then the fibrations for this model structure are simply the surjective maps. In the general case, there seems to be no easy characterization of the fibrant objects, and the model structure is practically useless for computations. In any case, we shall denote the underlying ∞-category by $\mathcal{DGA}_R$.

- One can consider an ∞-category $\mathcal{ER}_{R/}$ of E_∞-R-algebras. This is a slice ∞-category of a larger ∞-category $\mathcal{ER}$ of E_∞-ring spectra. As with A_∞-ring spectra, there are multiple ways of defining $\mathcal{ER}$. One can consider spectra equipped with the structure of an algebra over an appropriate E_∞-operad, or one can consider strictly commutative and associative monoid objects in an appropriate symmetric monoidal model category of spectra. Every ordinary commutative ring may be regarded as an E_∞-ring spectrum, and one can then define $\mathcal{ER}_{R/}$ as the ∞-category of objects $A \in \mathcal{ER}$ equipped with a map $R \to A$.

In general, we have functors $\mathcal{SCR}_{R/} \xrightarrow{\phi} \mathcal{DGA}_R \xrightarrow{\psi} \mathcal{ER}_{R/}$. If R is a $\mathbf{Q}$-algebra, then ψ is an equivalence of ∞-categories, ϕ is fully faithful, and the essential image of ϕ consists of the connective objects of $\mathcal{DGA}_R \simeq \mathcal{ER}_{R/}$ (that is, those algebras A having $\pi_i A = 0$ for $i < 0$). If R is not a $\mathbf{Q}$-algebra, then neither ϕ nor ψ nor $\psi \circ \phi$ is fully faithful and the situation is much more complicated. In this case, the ∞-category $\mathcal{DGA}_R$ is poorly behaved: it is both difficult to compute with (for reasons explained above), and conceptually unsuitable because it is not clear what notion it is intended to model. However, both $\mathcal{SCR}_{R/}$ and $\mathcal{ER}_{R/}$ have conceptual interpretations:

- Connective objects of $\mathcal{ER}$ are spaces equipped with addition and multiplication laws which are commutative, associative, and distributive up to coherent homotopy.

- Objects $\mathcal{SCR}$ are spaces equipped with addition and multiplication laws which are commutative, associative and distributive on the nose: that is, they are topological commutative rings.

From this point of view, the functor $\theta = \psi \circ \phi : \mathcal{SCR} \to \mathcal{ER}$ is easy to understand. We note that there are two obvious reasons why θ cannot be an equivalence: first, the initial object of $\mathcal{SCR}$ is the ordinary commutative ring $\mathbf{Z}$, so that the essential image of θ consists entirely of $\mathbf{Z}$-algebras (in contrast, the initial object of $\mathcal{ER}$ is the sphere spectrum, which is very nondiscrete: its homotopy groups are the stable homotopy groups of spheres). Second, the essential image of θ consists only of connective objects. But θ is not even an equivalence onto the ∞-category $\mathcal{ER}^c_{\mathbf{Z}/}$ of connective $\mathbf{Z}$-algebras. Objects of $\mathcal{ER}^c_{\mathbf{Z}/}$ may be thought of as topological spaces equipped with a strictly commutative addition, and a multiplication which is commutative (and distributive over addition) up to all higher homotopies. It turns out that, in contrast to the associative case, this is substantially weaker than the requirement of a strictly commutative multiplication.

So we are faced with two plausible candidates for our theory of "generalized rings": E_∞-ring spectra and simplicial commutative rings. Which is the better notion? The answer depends, of course, on what we want to do. The notion of an E_∞-ring spectrum is extremely useful in stable homotopy theory. Having observed that the complex K-theory $K(X)$ of any space X has a commutative ring structure, one would like to explain this by saying that, in some sense, K-theory itself is a commutative ring. The theory of E_∞-ring spectra provides the correct language for describing the situation: K-theory and many other generalized cohomology theories of interest may be endowed with E_∞-structures.

While the notion of an E_∞-ring spectrum is useful for applying algebraic ideas to homotopy theory, simplicial commutative rings seem better suited for the dual purpose of bringing homotopy theoretic ideas into algebra. If we take the point of view that our ultimate interest is in *ordinary* commutative rings, but some constructions such as (left derived) tensor products seem to force more general objects upon us, then the ∞-category $\mathcal{SCR}$ seems better suited to our needs: it is fairly conservative generalization of the notion of a commutative ring, yet sufficiently general for our purposes.

It is our opinion that the theory of simplicial commutative rings provides the appropriate notion of "generalized ring" for use in derived algebraic geometry. Here are some advantages of this choice:

- It does not seem appropriate to employ nonconnective ring spectra in constructing the basic "affine building blocks" of derived algebraic geometry. If we were working with E_∞-ring spectra, we would need to restrict our attention to the connective objects. However, in $\mathcal{SCR}$ the connectivity condition is automatically satisfied. (We note, however, that in characteristic zero the use of nonconnective algebras leads to a good notion of *weakly affine* algebraic stacks which includes, for example, the classifying stack for a unipotent algebraic group (see [35]). However, it is our opinion that non-connective E_∞-ring spectra do not provide the correct approach to this notion in positive characteristic. We will discuss a version of this notion in [23].)

- Objects of $\mathcal{SCR}$ are much easier to describe and to compute with than objects in $\mathcal{ET}$, or even $\mathcal{ET}_{\mathbf{Z}/}$.

- Though it is possible to set up the formal aspects of the theory of algebraic geometry in the context of E_∞-ring spectra, it seems that many constructions of algebro-geometric interest cannot be carried out in this setting. For example, we do not know how to define an analogue of the algebraic group SL_2 over the sphere spectrum. The definition seems to require the existence of a determinant for a rank 2-module, and it precisely the existence of these kinds of constructions which distinguishes $\mathcal{SCR}$ from $\mathcal{ET}$ (see Remark 2.6.5).

- In classical algebraic geometry, the affine line $\mathbf{A}^1_S$ over a scheme S is *flat* over S. This key basic fact fails in the E_∞-context, even if we assume that S is an ordinary scheme. This is because the "free" E_∞-$\mathbf{Z}$-algebra on one generator is not the ordinary ring $\mathbf{Z}[x]$. Instead, the appropriate free algebra R has

$$\pi_n(R) = \bigoplus_{m=0}^{\infty} \mathrm{H}_n(\Sigma_m, \mathbf{Z})$$

where the symmetric group Σ_m acts trivially on $\mathbf{Z}$. This ring spectrum is not flat over $\mathbf{Z}$ in any reasonable sense. Thus, if we were to employ E_∞-ring spectra (or even E_∞-$\mathbf{Z}$-algebras) in our foundations, then we would have to distinguish between the "flat affine line" $\operatorname{Spec} \mathbf{Z}[x]$ and the "additive group" $\operatorname{Spec} R$. In order to get any reasonable analogue of classical algebraic geometry, we need to force these two versions of the affine line to coincide.

We conclude this section by giving two more ways to think about the difference between E_∞-ring spectra and simplicial commutative rings. This may be safely omitted by the reader, since we will afterward have no need to consider E_∞-ring spectra at all.

As we remarked above, the functor θ factors through a functor $\theta' : \mathcal{SCR} \to \mathcal{ET}^c_{\mathbf{Z}/}$, where the superscript c indicates that we consider only connective E_∞-ring spectra. In concrete terms, the functor θ' "forgets" the strict commutativity of multiplication.

Proposition 2.6.1. *The functor θ' commutes with all limits and colimits.*

Proof. Simplicial commutative rings and connective E_∞-ring spectra both have "underlying spaces" such that a morphism is an equivalence if and only if it induces an equivalence on the underlying spaces. Moreover, θ' is compatible with the formation of these underlying spaces. The assertion concerning limits follows immediately from the fact that the formation of limits commutes with the formation of the underlying spaces in both cases.

For colimits, we must work a little harder. First of all, it suffices to prove the assertion for filtered colimits and finite colimits, since any colimit is a filtered colimit of finite colimits. For filtered colimits we can apply the same argument as above (since the formation of filtered colimits commutes with the formation of the underlying spaces). To prove that θ' commutes with finite colimits, it suffices to show that it preserves initial objects and pushouts. The initial object in both settings is the discrete ring $\mathbf{Z}$ which is preserved by θ'. Pushouts in both ∞-categories are given by (derived) tensor products. □

By the adjoint functor theorem, we see that θ' admits a right adjoint which we shall denote by θ_*. Using the adjoint functors θ' and θ_*, we may characterize the ∞-category $\mathcal{SCR}$ as an ∞-category of *coalgebras* over the *comonad* given by the adjunction $\theta'\theta_*$. Let $\mathcal{C}$ denote the ∞-category of coalgebras over this comonad.

Proposition 2.6.2. *The natural functor $\mathcal{SCR} \to \mathcal{C}$ is an equivalence of ∞-categories.*

Proof. This follows from the ∞-category version of the Barr-Beck theorem, since θ' commutes with all limits and detects equivalences. $\square$

Thus $\mathcal{SCR}$ is an ∞-category of coalgebras over $\mathcal{EJ}^c_{\mathbf{Z}/}$. In order to understand the relevant comonad, let us describe the functor θ_* more explicitly. The discrete ring $\mathbf{Z}[x]$ is the "free object on a zero cell" in the ∞-category $\mathcal{SCR}$. In other words, for any $R \in \mathcal{SCR}$, the underlying space of R is given by $\mathrm{Hom}_{\mathcal{SCR}}(\mathbf{Z}[x], R)$. Thus for any $S \in \mathcal{EJ}^c_{\mathbf{Z}/}$, the underlying space of $\theta_* S$ is given by

$$\mathrm{Hom}_{\mathcal{SCR}}(\mathbf{Z}[x], \theta_* S) = \mathrm{Hom}_{\mathcal{EJ}^c_{\mathbf{Z}/}}(\theta' \mathbf{Z}[x], S) = \mathrm{Hom}_{\mathcal{EJ}^c_{\mathbf{Z}/}}(\mathbf{Z}[x], S)$$

It follows that on the level of the underlying spaces, the functor θ_* and the comonad $\theta'\theta_*$ are given by the formula by $S \mapsto \mathrm{Hom}_{\mathcal{EJ}^c_{\mathbf{Z}/}}(\mathbf{Z}[x], S)$.

Remark 2.6.3. To see the ring structure on the space $\mathrm{Hom}_{\mathcal{SCR}}(\mathbf{Z}[x], S)$, note that the affine line $\mathrm{Spec}\, \mathbf{Z}[x]$ is a commutative ring object in the category of affine schemes. Because $\mathbf{Z}[x]$ is flat over $\mathbf{Z}$, coproducts of copies of $\mathbf{Z}[x]$ in the category of ordinary commutative rings agree with the corresponding coproducts in $\mathcal{EJ}_{\mathbf{Z}/}$. Consequently, $\mathbf{Z}[x]$ also has the structure of an "commutative ring object" in $\mathcal{EJ}^{op}_{\mathbf{Z}/}$.

Now it is crucial to remember that the discrete ring $\mathbf{Z}[x]$ is *not* free in $\mathcal{EJ}_{\mathbf{Z}/}$, so that $\mathrm{Hom}_{\mathcal{EJ}_{\mathbf{Z}/}}(\mathbf{Z}[x], S)$ is in general distinct from the underlying space of S. To understand the difference, we note that given any point $y \in \pi_0 S$, the commutativity of the product operation on S gives an action of the symmetric group Σ_n on the point y^n (and so, for example, a group homomorphism $\Sigma_n \to \pi_1 S$). For any point in the image of $\mathrm{Hom}_{\mathcal{EJ}_{\mathbf{Z}/}}(\mathbf{Z}[x], S)$, this symmetric group action is induced by the corresponding action of Σ_n on $x^n \in \mathbf{Z}[x]$, which canonically trivial since $\mathbf{Z}[x]$ is a discrete space.

By considering coalgebras over the comonad $\mathrm{Hom}(\mathbf{Z}[x], \bullet)$, we are essentially *forcing* $\mathbf{Z}[x]$ to corepresent "underlying space" functor. In other words, we may view $\mathcal{SCR}$ as the ∞-category obtained from $\mathcal{EJ}^c_{\mathbf{Z}/}$ by forcing the ordinary ring $\mathbf{Z}[x]$ to be the free $\mathbf{Z}$-algebra generated by x.

There is another way to understand the difference between $\mathcal{SCR}$ and $\mathcal{EJ}$, based on a sort of "Tannakian philosophy". Let R be an A_∞ ring spectrum. Then specifying R is equivalent to specifying the ∞-category $\mathcal{M}_R$ of left R-modules, together with the distinguished object R. From this data, we may reconstruct R as the endomorphisms of the distinguished object (or, equivalently, of the corepresentable "fiber functor" $\mathrm{Hom}(R, \bullet)$). From this point of view, we should understand additional structure on R as coming from additional structure on the ∞-category $\mathcal{M}_R$. For example, if R is an E_2-ring spectrum, then $\mathcal{M}_R$ has a coherently associative tensor product operation (with the distinguished object as the identity).

If R is an E_∞-ring spectrum, then the ∞-category $\mathcal{M}_R$ is equipped with a tensor structure which is coherently commutative and associative. Consequently, for any R-module M, one obtains an action of the symmetric group Σ_n on the n-fold tensor power $M^{\otimes n}$, and by taking a colimit one can form a module of coinvariants $M^{\otimes n}_{\Sigma_n}$ for this action.

Now suppose that $R \in \mathcal{SCR}$, and that M is a *connective* R-module. Then we can model the situation by choosing a simplicial commutative ring which represents R, and a cofibrant simplicial module which represents M. We can then apply the nth symmetric power functor degreewise, to obtain a simplicial module which we shall denote by $\mathrm{Sym}^n_R(M)$. One can show that Sym^n_R preserves weak equivalences (between cofibrant objects), and thus induces an endofunctor on the ∞-category of connective R-modules (it is the nonabelian left derived functor of the classical symmetric power functor).

One can construct a map $M^{\otimes n}_{\Sigma_n} \to \mathrm{Sym}^n_R(M)$. This map is an equivalence if R is a $\mathbf{Q}$-algebra, but not in general. There is no way to recover the functor Sym^n_R using only the tensor structure on the ∞-category of R-modules (which depends only on the underlying E_∞-ring spectrum structure on R); these functors depend on the structure of simplicial commutative ring on R.

The functor Sym^n_R is useful because it behaves like the classical symmetric power functor. For example, if M is a projective module, then $\mathrm{Sym}^n_R(M)$ is also free of the expected rank. By contrast, the module of coinvariants $(M^{\otimes n})_{\Sigma_n}$ is usually not projective. If R is discrete, then this module has higher homotopy groups which come from the homology of the symmetric group Σ_n.

Remark 2.6.4. Unlike the functor $M \mapsto M^{\otimes n}_{\Sigma_n}$, the functor $\mathrm{Sym}^n_R(M)$ is defined *a priori* only when M is connective. In [23], we will discuss a generalization in which M is not assumed connective: however, this extension is very strangely behaved (for example, it often has nonvanishing homotopy groups in all degrees, even when M is perfect).

Remark 2.6.5. Using the symmetric power functors Sym^n_R, we can also construct exterior powers $\bigwedge^n_R$ by setting $\bigwedge^n_R(M) = \mathrm{Sym}^n_R(M[1])[-n]$. We will discuss this at length in §3.1, where we show that this is equivalent to considering the nonabelian left derived functors of the "nth exterior power" functor. Consequently, $\bigwedge^n_R$ carries free modules to free modules of the expected rank. Using exterior powers, we can define determinants, and therefore algebraic groups such as SL_2. By contrast, there does not seem to be analogue of the algebraic group SL_2 in the setting of E_∞-ring spectra.

Chapter 3

Derived Rings

In this section, we develop the "derived commutative algebra" that will be needed in the remainder of this paper. We begin with a review of the theory of simplicial commutative rings in §3.1. In §3.2 we recall the construction of the cotangent complex of a morphism in $\mathcal{SCR}$, characterize it by a universal property. The next section, §3.3, contains a discussion of the role of the cotangent complex in classifying square-zero extensions.

Using the cotangent complex, we shall in §3.4 set up a theory of smooth and étale morphisms which generalizes the corresponding part of classical commutative algebra. In §3.5 we will discuss various other properties of modules and algebras and their interrelationships.

We will discuss a derived version of Grothendieck's theory of dualizing complexes in §3.6. Finally, in §3.7 we prove a derived version of Popescu's theorem on the smoothing of ring homomorphisms.

3.1 Simplicial Commutative Rings

Let $\mathcal{C}$ denote the (ordinary) category of simplicial commutative rings. The category $\mathcal{C}$ admits a Quillen model structure, where the weak equivalences and fibrations are those morphisms which are weak equivalences and fibrations on the underlying simplicial sets. This model structure is cofibrantly generated, and we shall denote the corresponding ∞-category by $\mathcal{SCR}$.

An equivalent way to arrive at $\mathcal{SCR}$ is to work with the category $\mathcal{C}'$ of (compactly generated) topological commutative rings. This category again admits a Quillen model structure, where the weak equivalences and fibrations are given by those maps which are weak equivalences and (Serre) fibrations on the underlying spaces. The formation of singular complexes and geometric realizations give rise to a Quillen equivalence between $\mathcal{C}$ and $\mathcal{C}'$, so that they model the same underlying ∞-category.

Let $A \in \mathcal{SCR}$ be any object. We may think of A as a topological space with the structure of a commutative ring. As a topological space, A has homotopy groups $\{\pi_i A\}_{i \geq 0}$. Here, we always take the base point at 0, the additive identity in A. The additive structure on A induces an abelian group structure on each homotopy group $\pi_i A$. This group structure agrees with the usual group structure if $i > 0$. A classical argument also shows that $\pi_1 A$ acts trivially on all of the higher homotopy groups of A.

The ring structure of A induces a multiplication on the homotopy groups of A, which is defined as follows. Let $x \in \pi_m A$, $y \in \pi_n A$. We may represent x and y by maps $[0,1]^m \to A$, $[0,1]^n \to A$, whose restriction to the boundary of the cubes are identically zero. Then $xy \in \pi_{n+m} A$ is represented by the product map $[0,1]^{m+n} \to A$. This product depends on an identification of $[0,1]^{m+n}$ with $[0,1]^m \times [0,1]^n$. We note that the natural identification $[0,1]^{m+n} \simeq [0,1]^m \times [0,1]^n \simeq [0,1]^n \otimes [0,1]^m \simeq [0,1]^{n+m}$ involves a permutation of coordinates and has degree $(-1)^{nm}$. Consequently, the product on $\pi_* A$ is not commutative but instead satisfies the graded-commutativity law $xy = (-1)^{nm} yx$. The homotopy groups may be assembled into a graded ring $\pi_* A = \oplus_{i \geq 0} \pi_i A$ which is commutative "in the graded sense". In particular, $\pi_0 A$ is an ordinary commutative ring and each $\pi_i A$ has the structure of a module over $\pi_0 A$.

Any ordinary commutative ring A may be regarded as an object of $\mathcal{SCR}$, by considering it as a topological ring with the discrete topology. This identification is harmless because the corresponding functor from commutative rings to $\mathcal{SCR}$ is fully-faithful. In fact, if $B \in \mathcal{SCR}$ and A is an ordinary commutative ring, then $\operatorname{Hom}_{\mathcal{SCR}}(B, A)$ is equivalent to the *discrete set* of ordinary ring homomorphisms from $\pi_0 B$ into A. In other words, the inclusion of ordinary commutative rings into $\mathcal{SCR}$ is right adjoint to the functor π_0.

More generally, for each $n \geq 0$ one can consider the full subcategory of $\mathcal{SCR}$ consisting of *n-truncated objects.* An object $A \in \mathcal{SCR}$ is n-truncated if $\pi_i A = 0$ for $i > n$. One can give an equivalent but more intrinsic formulation as follows: an object $A \in \mathcal{SCR}$ if $\pi_i \operatorname{Hom}_{\mathcal{SCR}}(B, A) = 0$ for $i > n$ (and any choice of basepoint). The full subcategory of n-truncated objects of $\mathcal{SCR}$ is a localization of $\mathcal{SCR}$. This follows from general theory (see for example [22]), but one can also directly construct truncation functors $\tau_{\leq n} : \mathcal{SCR} \to \mathcal{SCR}$ by forming coskeleta on level of simplicial sets. The latter approach to the definition gives additional information: the natural map $A \to \tau_{\leq n} A$ induces an isomorphism on homotopy groups in dimensions $\leq n$. We will call an object $A \in \mathcal{SCR}$ *discrete* if it is 0-truncated. In this case, A is equivalent to the ordinary commutative ring $\pi_0 A$.

If A is a fixed simplicial commutative ring, then the ordinary category of simplicial A-modules admits a Quillen model structure, where the weak equivalences and fibrations are those maps which are weak fibrations and equivalences on the underlying simplicial sets. (Once again, one can give a topological construction as well.) The underlying ∞-category will be denoted by $(\mathcal{M}_A)_{\geq 0}$. This ∞-category is presentable, has a zero object, and the suspension functor is fully faithful. By Proposition 2.2.3, the ∞-category $\mathcal{M}_A$ of infinite loop objects in $(\mathcal{M}_A)_{\geq 0}$ is stable and equipped with a t-structure having $(\mathcal{M}_A)_{\geq 0} \subseteq \mathcal{M}_A$ as the full subcategory of connective objects. We will refer to objects of $\mathcal{M}_A$ as *A-modules* and to objects of $(\mathcal{M}_A)_{\geq 0}$ as *connective A-modules.* They are the same thing as left modules (connective left modules) over the underlying A_∞-ring spectrum of A. However, the ∞-category $\mathcal{M}_A$ has extra structure when $A \in \mathcal{SCR}$. For example, we may ignore the distinction between left A-modules and right A-modules, and view the tensor product operation as $\mathcal{M}_A$-valued. This tensor product is associative and commutative up to coherent homotopy.

Since the model structure on simplicial commutative rings is cofibrantly generated, the ∞-category $\mathcal{SCR}$ is presentable. In fact, even more is true: $\mathcal{SCR}$ has enough compact objects, so that $\mathcal{SCR} = \mathrm{Ind}(\mathcal{SCR}^c)$ where $\mathcal{SCR}^c$ denotes the subcategory of compact objects in $\mathcal{SCR}$. In fact, $\mathcal{SCR}$ is generated by the ordinary ring $\mathbf{Z}[x]$ in the sense that every object in $\mathcal{SCR}$ can be constructed from copies of $\mathbf{Z}[x]$ using colimits. The object $\mathbf{Z}[x]$ co-represents the "underlying space" functor on $\mathcal{SCR}$, and the formation of the underlying space is compatible with filtered colimits. The corresponding statements are also true for the slice ∞-category of A-algebras for each $A \in \mathcal{SCR}$, provided that we replace $\mathbf{Z}[x]$ by $A[x] = A \otimes_{\mathbf{Z}} \mathbf{Z}[x]$.

Any object of $\mathcal{SCR}$ has an underlying A_∞-ring spectrum, which is a connective $\mathbf{Z}$-algebra. In particular, we may immediately import various notions from the theory of A_∞-ring spectra and their modules to the theory of simplicial commutative rings and their modules. We shall say that an A-module M is connective, discrete, flat, faithfully flat, free, projective, perfect, almost perfect, or of Tor-amplitude $\leq n$ if it has the same property when regarded as a left module over the underlying A_∞-ring spectrum of A. Similarly, we can speak of A-algebras B as being flat or faithfully flat, if they are flat or faithfully flat as left A-modules. We say that A is Noetherian or coherent if its underlying A_∞-ring spectrum is left Noetherian or left coherent.

There is an forgetful functor $G : \mathcal{SCR}_{A/} \to (\mathcal{M}_A)_{\geq 0}$, which ignores the algebra structure and remembers only the corresponding module structure. The functor G has a left adjoint $M \mapsto \mathrm{Sym}^*_A M$, which carries M to the "free A-algebra generated by M". The underlying A-module of $\mathrm{Sym}^*_A M$ is the direct sum $\oplus_{n \geq 0} \mathrm{Sym}^n_A M$, where the functors Sym^n_A are the nonabelian left derived functors of the symmetric powers as discussed in §2.6.

Using the functor Sym^*_A, one can obtain a better understanding of the way that A-algebras are built. We may think of $\mathrm{Sym}^*_A(A[n])$ as the "free A-algebra" obtained by attaching a free n-cell. More generally, given any homotopy class $x \in \pi_{n-1} A$, we may view x as classifying a map $\mathrm{Sym}^*_{\mathbf{Z}}(\mathbf{Z}[n-1]) \to A$ and form the tensor product $A' = A \otimes_{\mathrm{Sym}^*_{\mathbf{Z}}(\mathbf{Z}[n-1])} \mathbf{Z}$. In this case, we say that A' has been obtained from A by *attaching an n-cell*, with *attaching map* x. If $x = 0$, we obtain the free algebra $\mathrm{Sym}^*_A(A[n])$ discussed above. Any A-algebra B may be regarded as the result of a transfinite sequence of cell attachments. Moreover, if the map $A \to B$ is n-connected, then B can be constructed by attaching cells only in dimensions $> n$.

In view of the fact that any morphism may be obtained through successive cell attachments, the structure of the free algebras $\mathrm{Sym}^*_A M$ plays an important role in the theory. If M is free, then $\mathrm{Sym}^n_A M$ is free (of the expected rank if M is of finite rank). In general, $\mathrm{Sym}^n_A M$ is hard to describe. However, it is possible to give a description in the case where $M[-1]$ or $M[-2]$ is free, as we now explain. For further details we refer the reader to [16], p. 322.

Lemma 3.1.1. *Let $A \in \mathcal{SCR}$. The functor $M \mapsto (\mathrm{Sym}^n_A M[1])[-n]$ is the nonabelian left derived functor of the "nth exterior power" functor.*

Proof. For $A \in \mathcal{SCR}$, let T_A denote the nonabelian left derived functor of the nth exterior power. Then T_A is uniquely characterized by the following two properties:

- If $A_\bullet$ is a simplicial object of $\mathcal{SCR}$ with geometric realization A, and $M_\bullet$ is a connective $A_\bullet$-module with geometric realization M, then the natural map $|T_{A_\bullet} M_\bullet| \to T_A M$ is an equivalence.

- If A is discrete, and M is a free A-module, then $T_A M$ is naturally isomorphic to the (classical) nth exterior power of M over A, considered as a discrete A-module.

It is clear that the functor $M \mapsto (\mathrm{Sym}_A^n M[1])[-n]$ has the first of these properties, since functor Sym_A^n has this property. It suffices now to prove the second.

We first remark that if $B \in \mathcal{SCR}$ and $x \in \pi_1 B$, then $x^2 = 0 \in \pi_2 B$. Indeed, it suffices to check this in the universal case where $B = \mathrm{Sym}_{\mathbf{Z}}^*(Z[1])$. We may write $B = \mathbf{Z} \otimes_{\mathbf{Z}[x]} \mathbf{Z}$. Consequently, $\pi_i B = \mathrm{Tor}_i^{\mathbf{Z}[x]}(\mathbf{Z}, \mathbf{Z})$, which vanishes for $i > 1$. Returning to the case where B is general, we obtain a natural map from the nth exterior power of $\pi_1 B$ over $\pi_0 B$ into $\pi_n B$. In particular, let us suppose that A is discrete, M a discrete A-module, and $B = \mathrm{Sym}_A^*(M[1])$. We may then consider the composite $\phi_M : \bigwedge_A^n M \to \bigwedge_{\pi_0 B}^n \pi_1 B \to \pi_n B = \pi_n(\bigoplus \mathrm{Sym}_A^i M[1]) \to \pi_n \mathrm{Sym}_A^n M[1]$. To complete the proof, it will suffice to show that if M is free, then ϕ_M is an isomorphism and $\pi_j \mathrm{Sym}_A^n M[1] = 0$ for $j \neq n$. The result is obvious for $n \leq 1$, so we may as well suppose that $n \geq 2$.

Clearly it suffices to consider the case where M is of finite rank (the general case may be handled by passing to filtered colimits). In this case, we may work by induction on the rank of M. Using the equivalence

$$\mathrm{Sym}_A^n(M \oplus N)[1] \simeq \oplus_{i+j=n} \mathrm{Sym}_A^i M[1] \otimes_A \mathrm{Sym}_A^j N[1],$$

we may reduce to the case where $M \simeq A$. In this case, $\bigwedge_A^n M = 0$, so it suffices to prove that $\mathrm{Sym}_A^n(M[1]) = 0$. One checks by explicit computation that the natural map $A \oplus M[1] \to \mathrm{Sym}_A^* M[1]$ is an equivalence, so that all other summands $\mathrm{Sym}_A^n(M[1])$ must vanish. $\square$

In order to state the next lemma, we must recall a bit of algebra. Let A be a (discrete) commutative ring, and M a (discrete) A-module. Then one may speak of the *divided power algebra* of M over A, denoted by $\Gamma_A M$. This is the free commutative A-algebra generated by symbols $m^{(n)}$ for $m \in M$ and $n \geq 0$, subject to the following relations:

- For each $m \in M$, $m^{(0)} = 1$.

- For each $m, m' \in M$, we have

$$(m + m')^{(n)} = \sum_{i+j=n} m^{(i)} m'^{(j)}.$$

- For each $m \in M$, $a \in A$, we have $(am)^{(n)} = a^n m^{(n)}$.

- For each $m \in M$, we have $m^{(i)} m^{(j)} = m^{(i+j)} \frac{(i+j)!}{i!j!}$.

The intuition is that the symbol $m^{(i)}$ represents the "divided power" $\frac{m^i}{i!}$. In characteristic zero, the division by $i!$ is legal and the preceding formula defines an isomorphism between $\Gamma_A M$ and the symmetric algebra $\mathrm{Sym}^*_A M$.

The ring $\Gamma_A M$ admits a unique grading such that $m^{(n)}$ is of degree n for all $m \in M$. We let $\Gamma^n_A M$ denote the nth graded piece of $\Gamma_A M$. If M is free, then $\Gamma^n_A M$ is also free. If M is free of finite rank, then $\Gamma^n_A M^\vee$ is naturally isomorphic to the dual to $\mathrm{Sym}^n_A M$ (to see this, one can realize $\Gamma_A M^\vee$ as an algebra of differential operators acting on $\mathrm{Sym}^*_A M$). Consequently, the surjective map $M^{\otimes n} \to \mathrm{Sym}^n_A M$ induces an injective map $\Gamma^n_A M^\vee \to (M^\vee)^{\otimes n}$, which realizes $\Gamma^n_A M^\vee$ as the module of invariants for the action of the symmetric group Σ_n on $(M^\vee)^{\otimes n}$. Passing to filtered colimits, we deduce that $\Gamma^n_A M$ may be identified with the invariant submodule of $M^{\otimes n}$ for any flat A-module M.

Lemma 3.1.2. *Let $A \in \mathcal{SCR}$. The functor $M \mapsto (\mathrm{Sym}^n_A M[2])[-2n]$ is naturally equivalent to the nonabelian left derived functor of the divided power functor Γ^n.*

Proof. As in the proof of Lemma 3.1.1, it suffices to show that if A is discrete and M is free, then $(\mathrm{Sym}^n_A M[2])[-2n]$ is naturally equivalent to the discrete A-module $\Gamma^n_A M$. Using Lemma 3.1.1, we obtain an equivalence $(\mathrm{Sym}^n_A M[2])[-2n] \simeq T_A(M[1])[-n]$, where T_A is the nonabelian left derived functor of $\bigwedge^n_A$.

We now compute $T_A(M[1])$ using a particular representation of $M[1]$ as a simplicial A-module. Namely, we consider the complex of discrete A-modules which is given by M in degree 1 and zero elsewhere, and let $N_\bullet$ be the simplicial (discrete) A-module which is associated to this complex by the Dold-Kan correspondence. In particular, N_k is isomorphic to the direct sum of k copies of M. It follows that $T_A(M[1])$ is represented by the simplicial A-module $P_\bullet = \bigwedge^n_A N_\bullet$. Now

$$P_k \simeq \oplus_{n=n_1+\ldots+n_k} \bigwedge_A^{n_1} M \otimes \ldots \otimes \bigwedge_A^{n_k} M.$$

In particular, P_n contains a summand isomorphic to $M^{\otimes n}$. Each of the face maps $P_n \to P_{n-1}$ vanishes on the module of Σ_n-invariant tensors in $M^{\otimes n}$, so that we obtain a canonical map $\phi_M : \Gamma^n_A M \to \pi_n P_\bullet$. To complete the proof, it will suffice to show that ϕ_M is an isomorphism and that $\pi_i P_\bullet = 0$ for $i \neq n$.

As in the proof of Lemma 3.1.1, we may reduce to the case where M is of finite rank and work by induction on the rank of M. Breaking M up as a direct sum, we can reduce to the case where M is of rank 1. In this case, the summand of P_k corresponding to a decomposition $n = n_1 + \ldots + n_k$ vanishes unless each $n_i \leq 1$, and is naturally isomorphic to $M^{\otimes n}$ otherwise. Consequently, we deduce that $P_\bullet \simeq M^{\otimes n} \otimes_{\mathbf{Z}} Q''_\bullet$, where $P'_\bullet$ is a simplicial abelian group with Q''_k freely generated by the collection of surjective maps of simplices $\Delta_k \to \Delta_n$. Consequently, we have an exact sequence $0 \to Q'_\bullet \to Q_\bullet \to Q''_\bullet \to 0$, where $Q_\bullet$ computes the homology of Δ_n and $Q'_\bullet$ computes the homology of $\partial\,\Delta_n$. It follows that $\pi_i Q''_\bullet = \mathrm{H}_i(\Delta_n; \partial\,\Delta_n) = \mathrm{H}_i(S^n)$, which is $\mathbf{Z}$ when $i = n$ and zero otherwise. This proves that $\pi_i P_\bullet = 0$ for $i \neq n$, and one proves that ϕ_M is an isomorphism by an easy computation. □

The following connectivity estimate is key to later calculations:

Proposition 3.1.3. *Let $A \in \mathcal{SCR}$ and M an n-connected A-module for $n > 0$. Then $\mathrm{Sym}^m_A M$ is $(n + 2m - 2)$-connected.*

Proof. We have $(\mathrm{Sym}^m_A M)[-2m] = TM[-2]$, where T is the nonabelian left-derived functor of the functor of mth divided powers. It therefore suffices to show that $\Gamma(M[-2])$ has the same connectivity as $M[-2]$. This follows from the construction of left derived functors: if $M[-2]$ is k-connected, then it can be represented by a cofibrant simplicial module which is zero in degrees $\leq k$, and the same is true of $\Gamma(M[-2])$. □

Remark 3.1.4. If we replace divided powers by exterior powers, then we can use the same proof to show that $\mathrm{Sym}^m_A M$ is $(n + m - 1)$-connected provided that $n \geq 0$. For $m, n \geq 1$, this bound is weaker than the bound given by Proposition 3.1.3.

If $f : A \to B$ is a morphism in $\mathcal{SCR}$, we shall say that B is a *finitely presented* A-algebra if it lies in the smallest subcategory of $\mathcal{SCR}_{A/}$ which contains $A[x]$ and is stable under the formation of finite colimits. We shall say that B is a *locally finitely presented* A-algebra if it is a compact object of $\mathcal{SCR}_{A/}$; in other words, if the functor $\mathrm{Hom}_A(B, \bullet)$ commutes with filtered colimits. An A-algebra B is locally finitely presented if and only if it is a retract of a finitely presented A-algebra.

We will also need to discuss a somewhat weaker finiteness condition on A-algebras:

Proposition 3.1.5. *Let $A \in \mathcal{SCR}$, $n \geq 0$, and B an A-algebra. The following conditions are equivalent:*

1. *There exists a finitely presented A-algebra B' and an morphism $B' \to B$ of A-algebras which induces isomorphisms $\pi_m B' \simeq \pi_m B$ for $m \leq n$.*

2. *The functor $\mathrm{Hom}_A(B, \bullet)$ commutes with filtered colimits when restricted to the ∞-category of n-truncated B-modules.*

3. *(If A is Noetherian.) The A-algebra $\tau_{\leq n} B$ is Noetherian and $\pi_0 B$ is a finitely presented $\pi_0 A$-algebra in the category of ordinary commutative rings.*

Proof. The proof of the equivalence of (1) and (2) is analogous to that of Proposition 2.5.7, and the proof that (3) implies (1) is analogous to that of Proposition 2.5.10. Assume that A is Noetherian and (1) holds. Replacing B by B', we may suppose that B is finitely presented over A; it suffices to show that B is Noetherian. By the Hilbert basis theorem, $\pi_0 B$ is Noetherian. To complete the proof, it will suffice to show that each $\pi_i B$ is a finitely generated module over $\pi_0 B$.

Working by induction on the number of cells, we can reduce to the case where B is obtained from A by attaching a k-cell. If $k = 0$, then the result is obvious. Otherwise, we have $B = A \otimes_{\mathrm{Sym}^*_{\mathbf{Z}} \mathbf{Z}[k-1]} \mathbf{Z}$. Consequently, there is a spectral sequence converging to the homotopy groups of B with $E_2^{p\bullet}$-term given by $\mathrm{Tor}_p^{\pi_\bullet \mathrm{Sym}^*_{\mathbf{Z}} \mathbf{Z}[k-1]}(\mathbf{Z}, \pi_\bullet A)$. In particular, $\pi_n B$

admits a finite filtration whose successive quotients are all subquotients of graded pieces of the Tor-groups described above. It is easy to see that each of these graded Tor-groups is a finitely generated module over $\pi_0 A$ in each degree. □

If the equivalent conditions of Proposition 3.1.5 are satisfied, then we shall say that B is *of finite presentation to order* n as an A-algebra, or that the morphism $A \to B$ is *of finite presentation to order* n. If B is of finite presentation to order n for all $n \gg 0$, then we shall say that B is *almost of finite presentation* over A.

We conclude this section with a few remarks concerning the relationship between the ∞-categories $\mathcal{M}_A$ and $\mathcal{SCR}_{A/}$, for $A \in \mathcal{SCR}$.

The adjunction between G and Sym_A^* gives rise to a monad on $(\mathcal{M}_A)_{\geq 0}$ and a comonad on $\mathcal{SCR}_{A/}$. Using the ∞-categorical Barr-Beck theorem, one can easily check that $\mathcal{SCR}_{A/}$ is equivalent to the ∞-category of $G\,\mathrm{Sym}_A^*$-modules in $\mathcal{M}_A$. Thus, we may regard the ∞-category of A-algebras as determined by the ∞-category of A-modules together with its theory of symmetric powers. On the other hand, we shall show in a moment that the ∞-category of A-algebras determines the ∞-category of A-modules in a much more direct manner.

As in the classical case, the algebra $\mathrm{Sym}_A^* M$ has a natural grading. If we ignore all of the graded pieces except for the first two, we obtain an A-algebra which we shall denote by $A \oplus M$. This A-algebra naturally retracts onto A, so that we may view it as an object of the ∞-category $\mathcal{SCR}_{A//A}$ of *augmented A-algebras*. This ∞-category possesses a zero object, namely A. We shall denote this object by $A \oplus M$. If we choose models in which A is a topological ring and M a topological A-module, then $A \oplus M$ can be modelled by the topological ring with underlying group $A \times M$ and multiplication given by $(a,m)(a',m') = (aa', am' + a'm)$.

We may imagine that the data of a connective A-module M is more or less equivalent to the specification of the augmented A-algebra $A \oplus M$. After all, from $A \oplus M$ we may recover M as the cokernel of the structure homomorphism $A \to A \oplus M$, or as the kernel of the augmentation map. To get a more precise statement, we should notice that the algebra $A \oplus M$ is not arbitrary, but comes equipped with additional structure reflecting the stability of $\mathcal{M}_A$. Since M has a delooping $M[1]$ in $\mathcal{M}_A$, we obtain a delooping $A \oplus M[1]$ of $A \oplus M$ in the ∞-category of augmented A-algebras. Iterating this procedure, we see that $A \oplus M$ is an "infinite loop" object in the ∞-category of augmented A-algebras. We will show that the converse of this observation holds: any such infinite loop object has the form $A \oplus M$.

To approach the problem systematically, we note that the construction of $A \oplus M$ from M defines a functor $\phi : (\mathcal{M}_A)_{\geq 0} \to \mathcal{SCR}_{A//A}$. The functor ϕ commutes with all limits, and therefore induces a functor between the ∞-categories of infinite loop objects which we shall denote by $\widetilde{\phi}$.

Theorem 3.1.6. *The functor $\widetilde{\phi}$ is an equivalence of ∞-categories.*

Proof. The proof requires some facts about the cotangent complex which will be established in the next section. We only sketch the proof; this result will not be needed later.

Let $\mathcal{C}_A$ denote the ∞-category of infinite loop objects in $\mathcal{SCR}_{A//A}$. We first note that the functor $\psi : \widetilde{A} \mapsto \ker(\widetilde{A} \to A)$ maps $\mathcal{SCR}_{A//A}$ to $(\mathcal{M}_A)_{\geq 0}$. There is a natural identification

$\psi \circ \phi$ with the identity on $(\mathcal{M}_A)_{\geq 0}$ and the functor ψ is compatible with all limits, so that ψ induces an exact functor $\widetilde{\psi} : \mathcal{C}_A \to \mathcal{M}_A$ which is left inverse to $\widetilde{\phi}$.

We first claim the following: let $C \in \mathcal{C}_A$ be such that $\widetilde{\psi}(C) \in \mathcal{M}_A$ is connective, and let $M \in \mathcal{M}_A$. Then $f_{C,M} : \mathrm{Hom}_{\mathcal{C}_A}(C, \widetilde{\phi}M) \to \mathrm{Hom}_{\mathcal{M}_A}(\widetilde{\psi}C, \widetilde{\psi}\widetilde{\phi}M) = \mathrm{Hom}_{\mathcal{M}_A}(\widetilde{\psi}C, M)$ is an equivalence. Since both sides are compatible with limits in M, it suffices to prove this when M is k-truncated. View C as an infinite loop object $\{\widetilde{A}_i, \widetilde{A}_i \simeq \Omega \widetilde{A}_{i+1}\}_{i \geq 0}$ in $\mathcal{SCR}_{A//A}$. Then the mapping space on the left is given by the inverse limit of the sequence of spaces

$$\mathrm{Hom}_{\mathcal{SCR}_{A//A}}(\widetilde{A}_i, A \oplus \tau_{\geq 0} M[i]) = \mathrm{Hom}_{\mathcal{M}_A}(L_{\widetilde{A}_i/A} \otimes_{\widetilde{A}_i} A, \tau_{\geq 0} M[i]).$$

Let K_i denote the cokernel of $\widetilde{A}_i \to A$. The connectivity assumption on C implies that K_i is $(i-1)$-connected. Consequently, $K_i \otimes_A A_i$ is $(i-1)$-connected so that the natural map $K_i \otimes_A A_i \to L_{\widetilde{A}_i/A}$ has $(2i)$-connected cokernel. Consequently, the natural map $K_i \to L_{\widetilde{A}_i/A} \otimes_{\widetilde{A}_i} A$ has $(2i)$-connected cokernel. For $i > k$, this implies that the natural map $\mathrm{Hom}_{\mathcal{M}_A}(L_{\widetilde{A}_i/A} \otimes_{\widetilde{A}_i} A, M[i]) \to \mathrm{Hom}_{\mathcal{M}_A}(K_i, \tau_{\geq 0} M[i])$ is an equivalence. Passing to the limit as $i \to \infty$, we deduce that $f_{C,M}$ is an equivalence.

We would like now to claim that $\widetilde{\psi}$ is left adjoint to $\widetilde{\phi}$. In other words, we would like to claim that $f_{C,M}$ is an equivalence for any $C \in \mathcal{C}_A$, and $M \in \mathcal{M}_A$. So far, we know that this is true whenever $\widetilde{\psi}C$ is connective. By shifting, we may deduce that $f_{C,M}$ is an equivalence whenever $\widetilde{\psi}C$ is $(-n)$-connected for $n \gg 0$. Passing to colimits, we deduce that $f_{C,M}$ is an equivalence whenever C can be obtained as the colimit of a sequence $C_0 \to C_1 \to \ldots$, where each C_i is $(-i)$-connected. In particular, this holds when take $C = \widetilde{\phi}M$ for $M \in \mathcal{M}_A$, with $C_i = \widetilde{\phi}(\tau_{\geq 1-i} M)$. This shows that $\widetilde{\phi}$ is fully faithful and that its essential image contains every $C \in \mathcal{C}_A$ which can be obtained as the colimit of a sequence $\{C_i\}$ where $\widetilde{\psi}C_i$ is $(-i)$-connected.

To complete the proof, it suffices to show that every object of $\mathcal{C}_A$ may be obtained as a colimit of a such a sequence. To see this, let $C \in \mathcal{C}_A$, and view C as an infinite loop space $\{\widetilde{A}_i\}$, where $\widetilde{A}_i \in \mathcal{SCR}_{A//A}$. Then each $\widetilde{A}_i$ may be viewed as an "E_∞-object" of $\mathcal{SCR}_{A//A}$; in other words, it has a coherently commutative and associative addition law. Consequently, we may use the constructions of [27] to produce arbitrarily many "connected" deloopings of $\widetilde{A}_i$, which together give an object $C_i[i] \in \mathcal{C}_A$. One may then compute $\widetilde{\psi}C_i = \tau_{\geq -i}\widetilde{\psi}C$. Moreover, since $\widetilde{\psi}$ detects equivalences, this computation also shows that C is the colimit of the sequence $\{C_i\}$. □

3.2 The Cotangent Complex

Let A be a commutative ring, B an A-algebra, and M a B-module. An *A-linear derivation from B to M* is an additive homomorphism $d : B \to M$ which annihilates A and satisfies the Leibniz rule $d(bb') = b'd(b) + bd(b')$. If A and B are fixed, then there exists a universal target for A-linear derivations. This is the module $\Omega_{B/A}$ of *Kähler differentials*, which is defined to be the free module generated by formal symbols $\{db\}_{b\in B}$, modulo the submodule of relations generated by $\{d(bb') - bd(b') - b'd(b)\}_{b,b'\in B}$ and $\{da\}_{a\in A}$.

If $\psi : B \to C$ is another morphism of commutative rings, then there is an exact sequence

$$\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0.$$

In general, this sequence is not exact on the left. This leads one to suspect that there exists some natural extension of the above short exact sequence to a long exact sequence, each term of which is a kind of left-derived functor of Ω. However, since Ω is a functor of commutative rings, and commutative rings do not form an abelian category, the notion of a left-derived functor needs to be interpreted in terms of the non-abelian homological algebra of Quillen. The motivating idea is that while $\Omega_{C/B}$ is in general badly behaved, it behaves well when C is a *free* B-algebra. In this case, $\Omega_{C/B}$ is a free C-module on a set of generators which may be taken in bijection with a set of generators for C over B, and for any A the above short exact sequence is exact on the left. Thus, we should imagine that free B-algebras are in some sense "acyclic" for the functor $\Omega_{\bullet/B}$, and try to use these to resolve arbitrary B-algebras. If C is an arbitrary B-algebra, we can replace C by a so-called "cofibrant resolution" $\widetilde{C}_\bullet \to C$, where $\widetilde{C}_\bullet$ is a simplicial commutative B-algebra each term of which is free, and its map to C is a weak equivalence of simplicial commutative rings. Then, applying $\Omega_{\bullet/B}$ termwise to $\widetilde{C}_\bullet$, we get a simplicial module over $\widetilde{C}_\bullet$, which may be interpreted as an object in the derived category of C-modules. This object is called the *cotangent complex* of C over B and is usually denoted by $L_{C/B}$. The 0th cohomology of the cotangent complex is the ordinary module of Kähler differentials, and the short exact sequence described above extends to an exact triangle

$$L_{B/A} \otimes^L_B C \to L_{C/A} \to L_{C/B} \to L_{B/A} \otimes^L_B C[1].$$

Of course, there is no real reason to require that A and B are ordinary commutative rings: the definition makes perfectly good sense when A and B are simplicial commutative rings to begin with. Moreover, in this setting the cotangent complex becomes much easier to understand, because it may be characterized by a universal property. We will take this universal property as our definition of the cotangent complex, and we will see that the above construction actually works to produce an object having this universal property.

Let us motivate the definition of the cotangent complex by reformulating the universal property of $\Omega_{B/A}$. We first note that giving an A-linear derivation from B into M is equivalent to giving an A-algebra section of the natural map $B \oplus M \to B$. The advantage of this description is that it mentions only modules, algebras, and the construction $(B, M) \mapsto B \oplus M$. We have all of these notions at our disposal in the derived setting, and may therefore attempt the same definition.

For later applications, it will pay to work a little bit more generally. Rather than considering the cotangent complex of a map of simplicial commutative rings $A \to B$, we will instead consider natural transformations $\mathcal{F} \to \mathcal{F}'$, where $\mathcal{F}, \mathcal{F}' : \mathcal{SCR} \to \mathcal{S}$ are $\mathcal{S}$-valued "moduli functors" on $\mathcal{SCR}$. We recover the classical situation by taking $\mathcal{F}$ and $\mathcal{F}'$ to be the corepresentable functors $\mathrm{Hom}_{\mathcal{SCR}}(B, \bullet)$ and $\mathrm{Hom}_{\mathcal{SCR}}(A, \bullet)$.

We first introduce the appropriate replacement for the notion of a B-module in the above setting. Let $\mathcal{F} : \mathcal{SCR} \to \mathcal{S}$ be a functor. A *quasi-coherent complex* M *on* $\mathcal{F}$ assigns to each $A \in \mathcal{SCR}$ and to each $\eta \in \mathcal{F}(A)$ an A-module $M(\eta)$, which varies functorially in η in the strict sense that there exists a coherent family of equivalences

$$M(\eta) \otimes_A A' \simeq M(\eta')$$

whenever $\eta' = \phi_* \eta$ for some $\phi : A \to A'$.

Remark 3.2.1. Since $\mathcal{SCR}$ is not a small ∞-category, the ∞-category $\mathrm{QC}_{\mathcal{F}}$ is in general a "very large" ∞-category: it may be that its morphism spaces are not set-sized. This issue will never arise for moduli functors $\mathcal{F}$ which are of interest to us.

Proposition 3.2.2. *Let* $\mathcal{F} : \mathcal{SCR} \to \mathcal{S}$ *be a functor.*

- *The quasi-coherent complexes on* $\mathcal{F}$ *form a stable* ∞*-category* $\mathrm{QC}_{\mathcal{F}}$.
- *If the functor* $\mathcal{F}$ *is accessible, then* $\mathrm{QC}_{\mathcal{F}}$ *is presentable.*

Proof. The ∞-category $\mathrm{QC}_{\mathcal{F}}$ is stable because it is a limit of stable ∞-categories. The second claim follows from straightforward cardinality estimation. □

Remark 3.2.3. Given any property P of modules which is stable under base change (see §3.5), we shall say that a quasi-coherent complex M on $\mathcal{F}$ has the property P if and only if each $M(\eta)$ has the property P, for $\eta \in \mathcal{F}(A)$.

Example 3.2.4. Suppose that $\mathcal{F}$ is the co-representable functor $\mathrm{Hom}(A, \bullet)$. Then giving a quasi-coherent complex M on $\mathcal{F}$ is equivalent to giving the A-module $M(\eta)$, where $\eta \in \mathcal{F}(A)$ is the universal element. Moreover, M has a property P (assumed to be stable under base change) if and only if $M(\eta)$ has the same property P when considered as an A-module.

For technical reasons, we need to introduce the following condition:

Definition 3.2.5. Let $A \in \mathcal{SCR}$, and let M be an A-module. Then M is *almost connective* if $M[n]$ is connective for $n \gg 0$.

Remark 3.2.6. Some authors use the term connective to refer to the property that we have called almost connective.

Remark 3.2.7. The property of being almost connective is stable under base change, so it also makes sense for quasi-coherent complexes. We remark that if M is an almost connective, quasi-coherent complex on $\mathcal{F}$, then there need not exist any value of n for which $M[n]$ is connective; n is required to exist only locally.

Let $\mathcal{F} : \mathcal{SCR} \to \mathcal{S}$ be a functor. Fix $C \in \mathcal{SCR}$ and $\eta \in \mathcal{F}(C)$, and consider the functor which assigns to each connective C-module M the fiber $\Omega(C, \eta, M)$ of $\mathcal{F}(C \oplus M) \to \mathcal{F}(C)$ over the point η. Often the functor $M \mapsto \Omega(C, \eta, M)$ is corepresentable by an almost connective object $L_{\mathcal{F}}(\eta) \in \mathcal{M}_C$. In this case, $L_{\mathcal{F}}(\eta)$ is covariantly functorial in η in the weak sense that a map $\psi : C \to C'$ induces a map

$$\phi_\psi : L_{\mathcal{F}}(\eta) \otimes_C C' \to L_{\mathcal{F}}(\psi_* \eta).$$

If ϕ_ψ is an equivalence for any ψ, then $L_{\mathcal{F}}$ is a quasi-coherent complex on the functor $\mathcal{F}$ which we shall call the *absolute cotangent complex* of $\mathcal{F}$.

Remark 3.2.8. A priori, it is not clear that a module $L_{\mathcal{F}}(\eta)$ co-representing the functor $\Omega(C, \eta, \bullet)$ is uniquely determined, since the $\Omega(C, \eta, \bullet)$ is only defined for connective C-modules. However, any object of $(\mathcal{M}_C)_{\geq -k}$ is uniquely determined by the functor that it corepresents on the subcategory $(\mathcal{M}_C)_{\geq 0}$. This follows from the formula $\mathrm{Hom}_{\mathcal{M}_C}(M, N) = \Omega^k \mathrm{Hom}_{\mathcal{M}_C}(M, N[k])$. This proves that a cotangent complex is uniquely determined, provided that each $L_{\mathcal{F}}(\eta)$ is almost connective. This condition will always be satisfied in practice.

The following property is immediate from the definition:

Proposition 3.2.9. *Suppose given a diagram $\{\mathcal{F}_\alpha\}_{\alpha \in \mathcal{I}}$ of functors $\mathcal{SCR} \to \mathcal{S}$, indexed by some small ∞-category $\mathcal{I}$. Suppose further that each $\mathcal{F}_\alpha$ has a cotangent complex $L_\alpha \in \mathrm{QC}_{\mathcal{F}_\alpha}$. Let $\mathcal{F}$ be the limit of the diagram, and define $L \in \mathrm{QC}_{\mathcal{F}}$ to be the colimit of the quasi-coherent complexes $L_\alpha | \mathcal{F}$. Then L is a cotangent complex for $\mathcal{F}$, provided that L is almost connective. (This last condition is always satisfied if, for example, the diagram is finite, or if each L_α is connective.)*

We will actually be more concerned with the case of *relative* cotangent complex $L_{\mathcal{F}/\mathcal{G}}$, associated to a natural transformation $p : \mathcal{F} \to \mathcal{G}$ of functors. If $\mathcal{F}$ and $\mathcal{G}$ have cotangent complexes, then we note that p induces a natural transformation $L_{\mathcal{G}} | \mathcal{F} \to L_{\mathcal{F}}$. We may then define $L_{\mathcal{F}/\mathcal{G}}$ to be the cokernel of this transformation. Alternatively, we note that the cokernel in question may be characterized by the following universal property: for any $\eta \in \mathcal{F}(C)$ and any connective C-module M, the space of maps $\mathrm{Hom}_{\mathcal{M}_C}(L_{\mathcal{F}/\mathcal{G}}(\eta), M)$ is given by the fiber of the map $\mathcal{F}(C \oplus M) \to \mathcal{F}(C) \times_{\mathcal{G}(C)} \mathcal{G}(C \oplus M)$. We take this latter property as the definition of the relative cotangent complex: it is sometimes the case that $L_{\mathcal{F}/\mathcal{G}}$ exists even when $L_{\mathcal{F}}$ and $L_{\mathcal{G}}$ do not.

The following functorial property of the relative cotangent complex follows easily from the definition:

Proposition 3.2.10. *Let $\mathcal{F} : \mathcal{SCR} \to \mathcal{S}$ be a functor with a cotangent complex $L_{\mathcal{F}/\mathcal{G}} \in \mathrm{QC}_{\mathcal{F}}$, and let $\mathcal{G}' \to \mathcal{G}$ be any natural transformation. Then $L_{\mathcal{F}/\mathcal{G}} | \mathcal{F}'$ is a cotangent complex for the projection $\mathcal{F}' = \mathcal{F} \times_{\mathcal{G}} \mathcal{G}' \to \mathcal{G}'$.*

The next property is slightly less obvious, and we will need the following lemma:

Lemma 3.2.11. *Let $A \in \mathcal{SCR}$, let $\mathcal{G}, \mathcal{G}', \mathcal{G}'' : (\mathcal{M}_A)_{\geq 0} \to \mathcal{S}$ be functors. Suppose that $\mathcal{G}'$ and $\mathcal{G}''$ are co-representable by almost connective objects $L', L'' \in \mathcal{M}_A$, and suppose there exists a fiber sequence*

$$\mathcal{G}' \to \mathcal{G} \to \mathcal{G}''.$$

Then $\mathcal{G}$ is co-representable by an almost connective object of $L \in \mathcal{M}_A$.

Proof. We remark that $\mathcal{G}''(M)$ has a natural base point for each M, given by the zero map $L'' \to M$; it is with respect to this base point that the fiber is taken. Similarly, $\mathcal{G}'(M)$ has a natural base point; this gives a natural base point of $\mathcal{G}(M)$. We may then extend the definition of $\mathcal{G}(M)$ to all $M \in (\mathcal{M}_A)_{\geq -n}$ by the formula

$$\mathcal{G}(M) = \Omega^n \mathcal{G}(M[n]).$$

This formula also shows that $\mathcal{G}(M)$ admits a functorial sequence of deloopings, so that we may view $\mathcal{G}$ as an object in the ∞-category $\mathcal{C}$ of spectrum-valued functors on almost connective objects of $\mathcal{M}_A$. The same reasoning gives extensions of $\mathcal{G}'$ and $\mathcal{G}''$. The image of the Yoneda embedding is a stable subcategory of $\mathcal{C}$. Since $\mathcal{G}'$ and $\mathcal{G}''$ belong to this subcategory, so does $\mathcal{G}$. Thus $\mathcal{G}$ is representable by an almost connective object of $\mathcal{M}_A$. □

Proposition 3.2.12. *Let $\mathcal{F} \to \mathcal{F}' \to \mathcal{F}''$ be a sequence of natural transformations of functors. Suppose that there exists a cotangent complex $L_{\mathcal{F}'/\mathcal{F}''}$. Then there is an exact triangle*

$$L_{\mathcal{F}'/\mathcal{F}''}|\,\mathcal{F} \to L_{\mathcal{F}/\mathcal{F}''} \to L_{\mathcal{F}/\mathcal{F}'}$$

in the sense that if either the second or third term exists, then so does the other and there is a triangle as above.

Proof. If $\mathcal{G} \to \mathcal{G}'$ is any transformation of functors, let us abuse notation by writing $\mathrm{Hom}_{\mathcal{M}_A}(L_{\mathcal{G}/\mathcal{G}'}(\eta), M)$ for the homotopy fiber of $\mathcal{G}(A \oplus M) \to \mathcal{G}(A) \times_{\mathcal{G}'(A)} \mathcal{G}'(A \oplus M)$, for any $A \in \mathcal{SCR}$, $M \in (\mathcal{M}_A)_{\geq 0}$, $\eta \in \mathcal{G}(A)$.

Given any $A \in \mathcal{SCR}$, $M \in (\mathcal{M}_A)_{\geq 0}$, and $\eta \in \mathcal{F}(A)$, there exists a fiber sequence of spaces

$$\mathrm{Hom}_{\mathcal{M}_A}(L_{\mathcal{F}/\mathcal{F}'}(\eta), M) \to \mathrm{Hom}_{\mathcal{M}_A}(L_{\mathcal{F}/\mathcal{F}''}(\eta), M) \to \mathrm{Hom}_{\mathcal{M}_A}(L_{\mathcal{F}'/\mathcal{F}''}(\eta'), M)$$

where η' denotes the image of η in $\mathcal{F}'(A)$. Consequently, if $L_{\mathcal{F}/\mathcal{F}''}(\eta)$ exists as an almost connective object of $\mathcal{M}_A$, then $L_{\mathcal{F}/\mathcal{F}'}(\eta)$ can be constructed as a cokernel of the natural map

$$L_{\mathcal{F}'/\mathcal{F}''}(\eta') \to L_{\mathcal{F}/\mathcal{F}''}(\eta).$$

Conversely, if $L_{\mathcal{F}/\mathcal{F}'}(\eta)$ is representable by an almost connective complex, then so is $L_{\mathcal{F}/\mathcal{F}''}(\eta)$ by Lemma 3.2.11. The compatibility with base change follows from the triangle. □

For $A \in \mathcal{SCR}$, we let $\operatorname{Spec} A$ denote the corepresentable functor $\mathrm{Hom}_{\mathcal{SCR}}(A, \bullet)$. In considering the relative cotangent complexes of corepresentable functors, we will often omit "Spec" from the notation. Thus, we write $L_{B/A}$ for $L_{\operatorname{Spec} B/\operatorname{Spec} A}$, $L_{\mathcal{F}/A}$ for $L_{\mathcal{F}/\operatorname{Spec} A}$, and so forth.

For a map $A \to B$ in $\mathcal{SCR}$, the existence of $L_{B/A}$ is guaranteed by Proposition 3.2.14 below. First, we need an easy lemma:

Lemma 3.2.13. *Let $A \in \mathcal{SCR}$, and let M be a connective A-module. Let $B = \mathrm{Sym}^*_A(M)$ be the free A-algebra generated by M. Then $L_{B/A}$ exists and is naturally equivalent to $M \otimes_A B$.*

Proof. Let C be a B-algebra and N a C-module. We must compute the mapping fiber of $\mathrm{Hom}_A(B, C \oplus N) \to \mathrm{Hom}_A(B, C)$. Since B is free, this is equivalent to the mapping fiber of $\mathrm{Hom}_{\mathcal{M}_A}(M, C \oplus N) \to \mathrm{Hom}_{\mathcal{M}_A}(M, C)$, which is just $\mathrm{Hom}_{\mathcal{M}_A}(M, N) \simeq \mathrm{Hom}_{\mathcal{M}_B}(M \otimes_A B, N)$. □

Proposition 3.2.14. *Let $f : A \to B$ be a morphism in $\mathcal{SCR}$. Then $L_{B/A}$ exists and is connective. Moreover, if B is finitely presented (of finite presentation to order n, locally of finite presentation, almost of finite presentation) over A, then $L_{B/A}$ is a finitely generated (perfect to order n, perfect, almost perfect) B-module.*

Proof. We first treat the case where $B = A[x]$. In this case, Lemma 3.2.13 implies that $L_{B/A}$ exists and is free on a single generator.

Any A-algebra B can be constructed from the A-algebra $A[x]$ by forming colimits. Consequently, the functor $\mathrm{Hom}_{\mathcal{SCR}}(B, \bullet)$ is a limit of functors having the form $\mathrm{Hom}_{\mathcal{SCR}}(A, \bullet)$. Using Proposition 3.2.9, we see that $L_{B/A}$ exists and is a colimit of copies of B. In particular, it is connective.

If B is finitely presented over A, then the above proof actually shows that $L_{B/A}$ is a *finite* colimit of copies of B. Therefore it is finitely presented. If B is of finite presentation to order n, then $\mathrm{Hom}_{\mathcal{SCR}_{A/}}(B, B \oplus M)$ commutes with filtered colimits in M for $M \in (\mathcal{M}_B)_{\leq n}$ so that $L_{B/A}$ is perfect to order n. The same argument applies if B is locally of finite presentation or almost of finite presentation to show that $L_{B/A}$ is perfect or almost perfect. □

Remark 3.2.15. Combining Propositions 3.2.10, 3.2.12, and 3.2.14, we deduce that a natural transformation $L_{\mathcal{F}/\mathcal{G}}$ has a cotangent complex if and only if, for any $B \in \mathcal{G}$, the functor $\mathcal{F}' = \mathcal{F} \times_{\mathcal{G}} \mathrm{Spec}\, B$ has an absolute cotangent complex $L_{\mathcal{F}'}$.

We note that the proof of Proposition 3.2.14 also shows why the cotangent complex, as we have defined it, can be computed using the nonabelian derived functor approach of Quillen. If $f : A \to B$ is a map of ordinary commutative rings, and we make a cofibrant replacement $\widetilde{B}_\bullet$ for B, then B is the geometric realization of the simplicial object $\widetilde{B}_\bullet$, so that $L_{B/A}$ should be the geometric realization of the simplicial B-module $L_{\widetilde{B}_\bullet/A}$. Since each $\widetilde{B}_n$ is a free A-algebra, we deduce from the argument of Proposition 3.2.14 that $L_{\widetilde{B}_n/A}$ is a free $\widetilde{B}_n$-module, and therefore equivalent to the corresponding module of Kähler differentials.

We next prove a kind of Hurewicz theorem for the cotangent complex.

Proposition 3.2.16. *Let $f : A \to B$ be a morphism in $\mathcal{SCR}$, and let K denote the cokernel of this morphism (in the ∞-category $\mathcal{M}_A$). Then there exists a natural map $\phi : K \otimes_A B \to L_{B/A}$. Moreover, if f is n-connected for $n \geq 0$, then ϕ is $(n+2)$-connected.*

Proof. Since $L_{B/A}$ is connective, the identity map from $L_{B/A}$ to itself classifies a universal derivation $d : B \to B \oplus L_{B/A}$. Let $z : B \to B \oplus L_{B/A}$ denote the zero section. Then $d - z : B \to L_{B/A}$ is a map of A-modules. Since $d|A = z|A$, $d - z$ factors naturally through K. Tensoring up to B, we obtain the morphism ϕ.

Now we shall prove the connectivity statement. Let M_f denote the cokernel of ϕ. We note the following properties of M_f:

- The formation of M_f is compatible with filtered colimits in B.
- If we are given two composable maps $A \xrightarrow{f} B \xrightarrow{g} C$, then we get an exact triangle of C-modules
$$M_f \otimes_B C \to M_{g\circ f} \to M_g.$$
- The formation of M_f is compatible with base change in A.

If K is n-connected, then we may view B as obtained from A by a transfinite process of attaching k-cells for $k > n$. In view of the above naturality properties for M_f, it will suffice to show that if B is obtained from A by attaching a k-cell, then M_f is $(k+2)$-connected. Moreover, since the connectivity of M_f is unaltered by the base change $A \to \pi_0 A$, we may assume that A is discrete.

Suppose that $k = 1$. Then $B = A \otimes_{\mathbf{Z}[x]} \mathbf{Z}$ for some attaching map $\mathbf{Z}[x] \to A$, classifying an element $a \in \pi_0 A$. Then, as an A-module, B is the cokernel of the map $A \xrightarrow{a} A$. It follows that K may be identified with $A[1]$, so that $K \otimes_A B \simeq B[1]$. On the other hand, the relative cotangent complex $L_{B/A}$ may also be identified with $B[1]$ (since we are attaching a 1-cell). It is not difficult to check that the map ϕ is an isomorphism in this case, so that $M_f = 0$.

Now suppose that $k > 1$. Since A is discrete, the attaching map for any k-cell must be zero. Thus $B \simeq \mathrm{Sym}^*_A(A[k])$. It follows that $K \simeq \oplus_{m>0} \mathrm{Sym}^m_A(A[k])$, so that $K \otimes_A B = \oplus_{m>0} \mathrm{Sym}^m_B B[k]$. Also, we have $L_{B/A} = A[k] \otimes_A B = B[k]$. The map ϕ sends $\mathrm{Sym}^1_B B[k]$ isomorphically onto $L_{B/A}$, so that we may identify the kernel $M_f[-1]$ of ϕ with $\oplus_{m\geq 2} \mathrm{Sym}^m_B B[k]$. Since $B[k]$ is $(k-1)$-connected, Proposition 3.1.3 ensures that $\mathrm{Sym}^m_B B[k]$ is $(k+2m-3)$-connected, so that $M_f[-1]$ is $(k+1)$-connected. Consequently, the cokernel M_f is $(k+2)$-connected. □

Corollary 3.2.17. *A morphism $f : A \to B$ in $\mathcal{SCR}$ is an equivalence if and only if f induces an isomorphism $\pi_0 A \to \pi_0 B$ and $L_{B/A} = 0$.*

Proof. The "only if" direction is clear. Suppose, conversely, that $\pi_0 A$ maps isomorphically onto $\pi_0 B$. Let K denote the cokernel of f. If f is not an equivalence, then $\pi_n K \neq 0$ for some $n \geq 0$; choose n as small as possible. Then $\pi_n(K \otimes_A B) \simeq \pi_n L_{B/A} = 0$. On the other hand, the group on the left may also be computed as the ordinary tensor product of $\pi_n K$ with $\pi_0 B$ over $\pi_0 A$. Since $\pi_0 A \simeq \pi_0 B$, we deduce that $\pi_n K = 0$, a contradiction. □

We conclude this section by remarking that Proposition 3.2.14 has a converse, which gives a handy criterion for recognizing A-algebras of finite presentation:

Proposition 3.2.18. *Let $f : A \to B$ be a morphism in $\mathcal{SCR}$. The following are equivalent:*

1. *B is almost of finite presentation (locally of finite presentation, of finite presentation) over A.*

2. *The ordinary commutative ring $\pi_0 B$ is finitely presented over $\pi_0 A$ in the usual sense, and $L_{B/A}$ is almost perfect (perfect, finitely presented).*

Proof. It is clear that (1) implies (2) in all cases. We must prove the converse. Suppose first that $\pi_0 B$ is finitely presented over $\pi_0 A$ and that $L_{B/A}$ is almost perfect. We must show that B is almost finitely presented over A. By lifting a finite presentation of $\pi_0 B$ over $\pi_0 A$, we may reduce to the case where $\pi_0 A \simeq \pi_0 B$. We prove, by induction on $n \geq 0$, that there exists a factorization $A \to A_n \to B$, where A_n is finitely presented over A and $\pi_i A_n \simeq \pi_i B$ for $i \leq n$.

For $n = 0$, there is nothing to prove. Now suppose $n > 0$. Without loss of generality we may replace A by A_{n-1}. Let K denote the cokernel of $A \to B$, so that K is $(n-1)$-connected. From Proposition 3.2.16, we deduce that the natural map $K \otimes_A B \to L_{B/A}$ is $(n+1)$-connected. Consequently, $\pi_n L_{B/A} \simeq \pi_n(K \otimes_A B) \simeq \pi_n K$, and $\pi_i L_{B/A} = 0$ for $i < n$. Since $L_{B/A}$ is almost perfect, its first nonvanishing homotopy group is finitely presented as a discrete $\pi_0 B$-module. Consequently, we deduce that $\pi_n K$ is finitely presented as $\pi_0 A$-module. Attaching finitely many free n-cells to A, we may reduce to the case where $\pi_n K = 0$. Now K is n-connected, so that the same argument given above shows that $\pi_{n+1} K$ is finitely generated as a $\pi_0 A$-module. Each generator gives rise to a homotopy class $x \in \pi_n A$, together with a nullhomotopy in B. Using this data, we may enlarge A by attaching finitely many $(n+1)$-cells to kill the kernel of $\pi_n A \to \pi_n B$. This completes the construction of A_n, and the proof of the proposition in the "almost finite presentation" case.

Suppose next that $L_{B/A} = M[n]$, where M is a projective B-module and $n > 1$. Then $\pi_0 B \simeq \pi_0 A$. Let K denote the cokernel of $A \to B$. Then $\pi_n K$ is a projective $\pi_0 B \simeq \pi_0 A$-module, so that we may find a projective A-module P and a morphism $P[n] \to K$ which induces an isomorphism on π_n. Then we have a natural map of A-modules $P[n-1] \to K[-1] \to A$ which vanishes after tensoring with B. Let $C = A \otimes_{\mathrm{Sym}^*_{\mathbf{Z}} P[n-1]} \mathbf{Z}$ denote the A-algebra obtained by killing $P[n-1]$. Now we have a factorization $A \to C \to B$. By construction, $L_{C/A} \otimes_A B \simeq L_{B/A}$ so that $L_{B/C} = 0$. Thus $B \simeq C$ so that B is locally of finite presentation as an A-algebra.

Now suppose that B is almost of finite presentation over A and that $L_{B/A}$ is perfect. Let K be the cokernel of $A \to B$ and let k denote the least integer such that $\pi_k K \neq 0$. Then $L_{B/A}[-k]$ is connective and perfect, hence of Tor-amplitude $\leq n$ for some $n \gg 0$. We work by induction on n. If $n = 0$, then $L_{B/A}$ is a shift of a projective module and we are done. Replacing A by a finitely presented A-algebra if necessary, we may suppose that $k > 1$. Choose a system of generators for $\pi_k K$, and let A' denote the finitely presented A-module obtained by killing those generators in $\pi_{k-1} A$. Then we obtain a map $A' \to B$ with k-connected cokernel. Moreover, it is easy to check that the Tor-amplitude of $L_{B/A'}[-k-1]$ is

$\leq n-1$, provided that $n > 0$. By the inductive hypothesis, B is locally of finite presentation as an A'-algebra, hence locally of finite presentation as an A-algebra.

If $L_{B/A}$ is actually finitely presented, then the proof is the same except that we eventually reduce to the case where $L_{B/A}[-k]$ is a free module. □

3.3 Small Extensions

From the derived point of view, the entire cotangent complex $L_{B/A}$ can be characterized by a universal property. The classical language is suitable only for discussing trivial square-zero extensions of the form $B \oplus M$ when B and M are discrete, so that the same universal property can only be used to characterize $\mathrm{Hom}(L_{B/A}, M)$ when M is discrete. This determines the truncation $\tau_{\leq 0} L_{B/A} \simeq \Omega_{B/A}$, and this universal property is sometimes used as the definition of the Kähler differentials. However, there does exist a classical interpretation for a slightly larger bit of the cotangent complex, namely $\tau_{\leq 1} L_{B/A}$. This interpretation may be given as follows: if A and B are ordinary commutative rings and M is a B-module, then the 1-truncated space $\mathrm{Hom}_{\mathcal{M}_B}(L_{B/A}, M[1])$ is equivalent to the classifying space for the groupoid of square-zero extensions of B by M (as A-algebras). This is usually stated on the level of connected components: isomorphism classes of square-zero extensions of B by M are classified by $\pi_{-1} \mathrm{Hom}_{\mathcal{M}_B}(L_{B/A}, M) = \mathrm{Ext}^1(L_{B/A}, M)$.

We would like to obtain a similar interpretation of $\pi_{-1} \mathrm{Hom}_{\mathcal{M}_B}(L_{B/A}, M)$ in the case where A, B, and M are not necessarily discrete. Morally, it seems clear that this group again classifies equivalence classes of square-zero extensions, provided that the latter concept is suitably defined. Unfortunately, in the derived setting it is difficult to say what a square-zero extension *is*. In order to avoid this problem, we will take the universal property of the cotangent complex as a *definition*:

Definition 3.3.1. Let $A \in \mathcal{SCR}$, let B be an A-algebra, and let M be a connective B-module. A *small extension of B by M over A* consists of the following data:

- An object $\widetilde{B} \in \mathcal{SCR}$.
- An A-algebra section s of the projection $B \oplus M[1] \to B$.
- An identification of $\widetilde{B}$ with the pullback $B \times_{B \oplus M[1]} B$, where B maps to $B \oplus M[1]$ via s and via the zero section.

Remark 3.3.2. Since the algebra $\widetilde{B}$ is determined by the section $s : B \to B \oplus M[1]$, we could instead simply define a small extension to be a section s as above. This makes it clear that small extensions are classified by $\mathrm{Hom}_{\mathcal{M}_B}(L_{B/A}, M[1])$. The point of the inefficient definition given above is that we wish to emphasize the algebra $\widetilde{B}$ as the "total space" of the extension.

We will abuse terminology and simply refer to $\widetilde{B}$ as a *small extension of B*. We note that $\widetilde{B}$ is naturally equipped with the structure of an A-algebra, a morphism to B, and that the kernel of the map $\widetilde{B} \to B$ may be identified with M as an A-module. We should emphasize that simply specifying $\widetilde{B}$, even together with this additional structure, does not determine the data of the small extension except in special cases. It is to these cases which we shall turn next.

Proposition 3.3.3. *Suppose that $B \in \mathcal{SCR}$ be k-truncated, and that $I \subseteq \pi_k B$ is a $\pi_0 B$-submodule. Then there exists a k-truncated simplicial commutative ring B/I such that for any k-truncated $A \in \mathcal{SCR}$, $\mathrm{Hom}_{\mathcal{SCR}}(B/I, A) \subseteq \mathrm{Hom}_{\mathcal{SCR}}(B, A)$ is the union of connected components corresponding to those morphisms $B \to A$ such that the induced map $\pi_k B \to \pi_k A$ vanishes on I. Moreover, we have $\pi_i(B/I) \simeq \pi_i B$ for $i \neq k$, $\pi_k(B/I) = (\pi_k B)/I$.*

Proof. Consider I as a discrete $\mathbf{Z}$-module. There is a natural map of $\mathbf{Z}$-modules from $I[k]$ into B, hence a map f of $\mathbf{Z}$-algebras from $\mathrm{Sym}^*(I[k])$ into B. Let C denote the pushout $\mathbf{Z} \coprod_{\mathrm{Sym}^*(I[k])} B$, where the map $\mathrm{Sym}^*(I[k]) \to \mathbf{Z}$ classifies the zero map from $I[k]$ into $\mathbf{Z}$. One easily checks that $\mathrm{Hom}_{\mathcal{SCR}}(C, A) \subseteq \mathrm{Hom}_{\mathcal{SCR}}(B, A)$ is the union of those connected components of maps which vanish on I, whenever A is k-truncated. Let B/I denote the truncation $\tau_{\leq k} C$. Then B/I has the appropriate mapping properties. A simple calculation shows that B/I has the expected homotopy groups. □

Remark 3.3.4. The universal mapping property of B/I immediately implies that any other B-algebra having the same homotopy groups (as a $\pi_* B$-algebra) is canonically equivalent to B/I.

In the situation Proposition 3.3.3, we shall say that B is a *square-zero extension of B/I by $I[k]$* if either $k > 0$, or $k = 0$ and $I^2 = 0$ in $B \simeq \pi_0 B$. Note that in either case, I has a (unique) B/I-module structure which induces its natural B-module structure.

Proposition 3.3.5. *Let $A \in \mathcal{SCR}$ and let B be a k-truncated A-algebra. Let $\mathcal{C}$ denote the ∞-category of pairs (M, s) where $M[-k] \in (\mathcal{M}_B)_0$ and $s \in \mathrm{Hom}_{\mathcal{M}_B}(L_{B/A}, M[1])$ classifies a small extension $\widetilde{B}$ of B.*

Then the functor $(M, s) \mapsto \widetilde{B}$ from $\mathcal{C}$ to $\mathcal{SCR}_{A//B}$ is fully faithful, and its essential image consists of the square-zero extensions of B by B-modules concentrated in degree k.

Proof. Let F denote the functor in question. It is easy to see that for any $(M, s) \in \mathcal{C}$, the algebra $F(M, s)$ is a square zero extension of B in the ∞-category of A-algebras. We next show that F is fully faithful. In other words, we must show that for any pair of objects (M, s), (M', s'), the natural map

$$\mathrm{Hom}_{\mathcal{C}}((M, s), (M', s')) \to \mathrm{Hom}_{\mathcal{SCR}_{A//B}}(F(M, s), F(M', s'))$$

is an equivalence. We note that both sides are compatibly fibered over the discrete space $\mathrm{Hom}_{\mathcal{M}_B}(M, M')$. It therefore suffices to show that F induces an equivalence on the fiber over any given homomorphism $f : M \to M'$.

The fiber Y_f of $\mathrm{Hom}_{\mathcal{C}}((M, s), (M', s'))$ over f is the space of paths from s to $s' \circ f$ in $\mathrm{Hom}_{\mathcal{M}_B}(L_{B/A}, M[1])$. Consequently, this space is either empty, or is a torsor for $\mathrm{Hom}_{\mathcal{M}_B}(L_{B/A}, M)$, depending on whether or not the difference $s - (f \circ s')$ vanishes in $\pi_0 \mathrm{Hom}_{\mathcal{M}_B}(L_{B/A}, M'[1])$.

Let us now compute the space $X = \mathrm{Hom}_{\mathcal{SCR}_{A//B}}(F(M,s), F(M',s'))$. This space is again fibered over the discrete space $\mathrm{Hom}_{\mathcal{M}_B}(M, M')$; we will denote the fiber over a homomorphism f by X_f. To compute X make use of the fact that $F(M',s') \simeq B \times_{B \oplus M[1]} B$. This implies that X is given by the fiber of the map $\mathrm{Hom}_{\mathcal{SCR}_{A/}}(F(M,s), B) \to \mathrm{Hom}_{\mathcal{SCR}_{A/}}(F(M,s), B \oplus M'[1])$ where the base point is taken over the composite map $\gamma : F(M,s) \to B \xrightarrow{s'} B \oplus M'[1]$. Both sides are compatibly fibered over $\mathrm{Hom}_{\mathcal{SCR}_{A/}}(F(M,s), B)$; thus we may identify X with the space of paths in $\mathrm{Hom}_{\mathcal{M}_{F(M,s)}}(L_{F(M,s)/A}, M'[1]) = \mathrm{Hom}_{\mathcal{M}_B}(L_{F(M,s)/A} \otimes_{F(M,s)} B, M'[1])$ which join the morphism $\alpha : F(M,s) \to B \xrightarrow{0} B \oplus M'[1]$ to the morphism $\beta : F(M,s) \to B \xrightarrow{s'} B \oplus M'[1]$.

We have an exact triangle $L_{F(M,s)/A} \otimes_{F(M,s)} B \to L_{B/A} \to L_{B/F(M,s)}$, which induces a fibration

$$i : \mathrm{Hom}_{\mathcal{M}_B}(L_{F(M,s)/A} \otimes_{F(M,s)} B, M'[1]) \to \mathrm{Hom}_{\mathcal{M}_B}(L_{B/F(M,s)}[-1], M'[1])$$

having fiber $\mathrm{Hom}_{\mathcal{M}_B}(L_{B/A}, M'[1])$. We note that $i(\alpha)$ and $i(\beta)$ both have trivial image in the space $\mathrm{Hom}_{\mathcal{M}_B}(L_{B/F(M,s)}[-1], M'[1])$, so that any path from α to β gives rise to a point in the space $Z = \mathrm{Hom}_{\mathcal{M}_B}(L_{B/F(M,s)}, M'[1])$. Since the cokernel of $F(M,s) \to B$ is equivalent to $M[1]$, it is k-connected. By Proposition 3.2.16, there is a natural $(k+2)$-connected morphism $M[1] \otimes_{F(M,s)} B \to L_{B/F(M,s)}$. Since $M'[1]$ is $(k+1)$-truncated, we deduce that

$$Y = \mathrm{Hom}_{\mathcal{M}_B}(L_{B/F(M,s)}, M'[1]) \simeq \mathrm{Hom}_{\mathcal{M}_B}(M[1] \otimes_{F(M,s)} B, M'[1]) \simeq \mathrm{Hom}_{\mathcal{M}_{F(M,s)}}(M[1], M'[1]),$$

which is equivalent to the discrete space of $\pi_0 B$-module homomorphisms from M to M'. Moreover, the map $X \to Z$ corresponds simply to the map $X \to \mathrm{Hom}_{\mathcal{M}_B}(M, M')$ considered above.

Let us now fix a homomorphism $f : M \to M'$, corresponding to a path joining $i(\alpha)$ and $i(\beta)$. The fiber X_f is nonempty if and only if the path p can be lifted to a path joining α to β. The obstruction to lifting such a path lies in component group of the fiber $\mathrm{Hom}_{\mathcal{M}_B}(L_{B/A}, M'[1])$ of i. A simple computation shows that this obstruction is simply given by $s - (f \circ s') \in \pi_0 \mathrm{Hom}_{\mathcal{M}_B}(L_{B/A}, M'[1])$, so that X_f is nonempty if and only if Y_f is nonempty. Supposing that X_f is nonempty, we note that since Z is discrete, X_f has the structure of a torsor for the loop space of $\mathrm{Hom}_{\mathcal{M}_B}(L_{B/A}, M'[1])$. Moreover, the induced map $Y_f \to X_f$ is a map of torsors, and therefore a homotopy equivalence.

It remains to show that F is essentially surjective: that is, every square-zero extension of B arises as a small extension. Let $C \in \mathcal{SCR}_{A//B}$ be a square-zero extension of B by $I[k]$, where $I \subseteq \pi_k C$. Let $D = C \otimes_{\mathrm{Sym}^*_A(I[k])} A$, so that we may identify B with $\tau_{\leq k} D$. We also note that $B \otimes_C D \simeq \mathrm{Sym}^*_B I[k+1]$ as B-algebras.

The exact triangle $C \to B \to I[k+1]$ of C-modules becomes split after tensoring with B, so that we get a decomposition $B \otimes_C B \simeq B \oplus (B \otimes_C I[k+1])$ as B-modules. In particular, the natural map $B \otimes_C D \to B \otimes_C B$ induces on π_{k+1} the map $I[k+1] \to (\pi_0(B \otimes_C I))[k+1]$. Since I is square-zero, this map is an isomorphism. Consequently, we see that $\tau_{\leq k+1}(B \otimes_C B)$ is equivalent *as a B-algebra* to $\tau_{\leq k+1} \mathrm{Sym}^*_B I[k+1] \simeq B \oplus I[k+1]$. Now, C equalizes the two natural maps from B to $B \times_C B$, and therefore also the two maps from B to $B \oplus I[k+1]$. Consequently, we obtain a map $C \to B \times_{B \oplus I[k+1]} B$, which is an equivalence (this can be checked by computing homotopy groups). □

This proposition is very useful because square-zero extensions exist in abundance. For any k-truncated $B \in \mathcal{SCR}$, we may view B as obtained by making k successive square zero extensions of the discrete ring $\pi_0 B$. In fact, there is a more canonical construction which works more generally.

Proposition 3.3.6. *Let $f : A \to B$ be a k-connected morphism of simplicial commutative rings, $k > 0$. Let $d : B \to B \oplus L_{B/A}$ denote the universal derivation, and let $\widetilde{B}$ denote the corresponding small extension of B over A. Then $\widetilde{f} : A \to \widetilde{B}$ is $(k+1)$-connected.*

Proof. Let K denote the cokernel of f and $\widetilde{K}$ the cokernel of $\widetilde{f}$. Then we have an exact triangle $\widetilde{K} \to K \to L_{B/A}$. To prove that $\widetilde{K}$ is $(k+1)$-connected, it suffices to show that the map $K \to L_{B/A}$ is $(k+2)$-connected. For this, it suffices to show that $K \to K \otimes_A B$ and $K \otimes_A B \to L_{B/A}$ are $(k+2)$-connected. The second map is $(k+2)$-connected by Proposition 3.2.16. On the other hand, the cokernel of $K \to K \otimes_A B$ is $K \otimes_A K$. Since K is k-connected, $K \otimes_A K$ is $(2k+1)$-connected. The proposition now follows, since $2k+1 \geq k+2$. □

Remark 3.3.7. Consequently, for any 1-connected morphism $f : A \to B$, we may view A as the inverse limit of a tower of small extensions of B whose homotopy groups converge to the homotopy groups of A. Note that f is 1-connected if and only if $\pi_0 A \simeq \pi_0 B$ and $\pi_1 f : \pi_1 A \to \pi_1 B$ is surjective. In particular, the natural map $A \to \pi_0 A$ is always 1-connected.

Thus, in some sense, understanding maps between arbitrary objects of $\mathcal{SCR}$ can be reduced to understanding maps between discrete commutative rings and understanding certain "linearized" mapping problems. The importance of the cotangent complex is that it controls these linearized mapping problems.

There is another situation in which we can give an explicit characterization of the space of small extensions: the case in which the kernel of the extension is induced from some module over the ground ring. More precisely, we have:

Proposition 3.3.8. *Let $A \in \mathcal{SCR}$, and let B be a A-algebra. Let M_A be a connective A-module, and set $M_B = M_A \otimes_A B$, $A' = A \oplus M_A$. Let $\mathcal{C} \subseteq \mathcal{SCR}_{A'//B}$ be the full subcategory consisting of those algebras B' for which the natural map $B' \otimes_{A'} A \to B$ is an equivalence. Then $\mathcal{C}$ is a small ∞-groupoid, and its classifying space is naturally equivalent to the space $\mathrm{Hom}_{\mathcal{M}_B}(L_{B/A}, M_B[1])$ of small extensions of B by M_B.*

Proof. Let B' be an algebra as above. Since $A \oplus M_A$ is given by the fiber product $A \times_{A \oplus M_A[1]} A$, by tensoring with B' over A' we deduce an equivalence $B' \simeq B \otimes_{B \oplus M_B} B$. Thus, B' is the total space of some small extension of B by M_B (over A). To complete the proof, we consider two such small extensions B' and B'' and compute the fiber of the map

$$\mathrm{Hom}_{A'}(B', B'') \to \mathrm{Hom}_{A'}(B', B).$$

Since $B'' \simeq B \otimes_{B \oplus M_B[1]} B$, we see that this fiber is also the fiber of the natural map

$$\mathrm{Hom}_{A'}(B', B) \to \mathrm{Hom}_{A'}(B', B \oplus M_B[1]).$$

Base changing from A' to A, we are studying the homotopy fiber of

$$\mathrm{Hom}_{A}(B, B) \to \mathrm{Hom}_{B}(B, B \oplus M_B[1]).$$

This is nonempty if and only if B' and B'' are equivalent as small extensions, and in this case is a torsor for $\mathrm{Hom}_{\mathcal{M}_B}(L_{B/A}, M_B)$ as desired. □

3.4 Smooth and Étale Morphisms

In this section, we will explain how to generalize the notion of smooth and étale ring homomorphisms to simplicial commutative rings. We begin with a few general remarks.

Let $\{A_\alpha\}$ be a diagram in $\mathcal{SCR}$ having limit A. If $\mathcal{F} : \mathcal{SCR} \to \mathcal{S}$ is any functor, then there is a natural map

$$\phi : \mathcal{F}(A) \to \lim \mathcal{F}(A_\alpha).$$

If $\mathcal{F}$ is a corepresentable functor given by $\mathcal{F}(R) = \mathrm{Hom}_{\mathcal{SCR}}(B, R)$ for some $B \in \mathcal{SCR}$, then ϕ is an equivalence for *any* diagram $\{A_\alpha\}$. If $\mathcal{F}$ is representable by a geometric object which is not affine, then it is unrealistic to expect that ϕ is an equivalence for arbitrary diagrams. However we should still expect that ϕ will be an equivalence in cases where the limiting algebra A has a geometric interpretation.

Definition 3.4.1. Let $\mathcal{F} : \mathcal{SCR} \to \mathcal{S}$ be a functor. We shall say that $\mathcal{F}$ is *nilcomplete* if, for any $A \in \mathcal{SCR}$, the natural map $\mathcal{F}(A) \to \lim\{\mathcal{F}(\tau_{\leq n} A)\}$ is an equivalence.

We shall say that $\mathcal{F}$ is *infinitesimally cohesive* if, for any $A \in \mathcal{SCR}$ and any small extension $\widetilde{A}$ of A by an A-module M, the natural map

$$\mathcal{F}(\widetilde{A}) \to \mathcal{F}(A) \times_{\mathcal{F}(A \oplus M[1])} \mathcal{F}(A)$$

is an equivalence.

We shall say that $\mathcal{F}$ is *cohesive* if for any pair $A \to C$, $B \to C$ of surjective morphisms in $\mathcal{SCR}$, the induced map $\mathcal{F}(A \times_C B) \to \mathcal{F}(A) \times_{\mathcal{F}(C)} \mathcal{F}(B)$ is an equivalence.

More generally, we shall say that a transformation $\mathcal{F} \to \mathcal{F}'$ is *nilcomplete* (*infinitesimally cohesive*, *cohesive*) if the fiber product $\mathcal{F} \otimes_{\mathcal{F}'} \mathrm{Spec}\, B$ is nilcomplete (infinitesimally cohesive, cohesive), for any $\eta \in \mathcal{F}'(B)$. Here we let $\mathrm{Spec}\, B$ denote the corepresentable functor $\mathrm{Hom}_{\mathcal{SCR}}(B, \bullet)$.

Remark 3.4.2. Let $\mathcal{F}$ be a functor $\mathcal{SCR} \to \mathcal{S}$. The condition that $\mathcal{F}$ be nilcomplete and infinitesimally cohesive mixes very well with the requirement that $\mathcal{F}$ have a cotangent complex. Suppose, for example, that $A \to B$ is a 1-connected morphism in $\mathcal{SCR}$, and we wish to study the fiber of $\mathcal{F}(A) \to \mathcal{F}(B)$. In this case, we may write A as the inverse limit of tower of successive small extensions B_n of B. Since the structure maps of this tower become highly connected, the nilcompleteness of $\mathcal{F}$ implies that $\mathcal{F}(A) \simeq \lim\{\mathcal{F}(B_n)\}$. Moreover, each B_{n+1} is an infinitesimal extension of B_n by some B_n-module M. The infinitesimal cohesiveness of $\mathcal{F}$ tells us that $\mathcal{F}(B_{n+1})$ can be computed in terms of $\mathcal{F}(B_n)$ and $\mathcal{F}(B_n \oplus M)$, and the relationship between these spaces is controlled by the cotangent complex of $\mathcal{F}$.

Definition 3.4.3. Let $T : \mathcal{F} \to \mathcal{F}'$ be a natural transformation of functors $\mathcal{F}, \mathcal{F}' : \mathcal{SCR} \to \mathcal{S}$. We shall say that T is

- *weakly formally smooth* if it has a relative cotangent complex $L_{\mathcal{F}/\mathcal{F}'}$ which is the dual of a connective, perfect complex.
- *formally smooth* if it is weakly formally smooth, nilcomplete, and infinitesimally cohesive.
- *formally étale* if it is formally smooth and $L_{\mathcal{F}/\mathcal{F}'} = 0$.

Remark 3.4.4. A transformation $T : \mathcal{F} \to \mathcal{F}'$ is formally étale if and only if it satisfies the following lifting property: for any small extension $\widetilde{B} \to B$, the natural map $\phi : \mathcal{F}(\widetilde{B}) \to \mathcal{F}(B) \times_{\mathcal{F}'(B)} \mathcal{F}'(\widetilde{B})$ is an equivalence.

The same reasoning shows that if T is formally smooth, then the map ϕ is always surjective. However, our definition of formal smoothness places a much stronger condition on the functor T: it asserts that the fiber of ϕ is under good control, and in some sense finite dimensional. We remark that this is not analogous to the standard definition of formal smoothness, which requires only the lifting property and not the finite dimensionality.

Proposition 3.4.5. *Let $T : \mathcal{F} \to \mathcal{G}$ be a formally smooth transformation of functors $\mathcal{SCR} \to \mathcal{S}$. Let $f : A \to B$ be a 1-connected morphism in $\mathcal{SCR}$. Then the natural map $\phi : \mathcal{F}(A) \to \mathcal{F}(B) \times_{\mathcal{G}(B)} \mathcal{G}(A)$ induces a surjection on connected components. If T is formally étale, then ϕ is an equivalence.*

Proof. Realize A as the inverse limit of a tower of small extensions of B. □

Corollary 3.4.6. *Let $\mathcal{F} \to \mathcal{G}$ be a formally étale transformation of functors $\mathcal{SCR} \to \mathcal{S}$. Let $f : A \to B$ induce an isomorphism $\pi_0 A \simeq \pi_0 B$. Then the natural map $\phi : \mathcal{F}(A) \to \mathcal{F}(B) \times_{\mathcal{G}(B)} \mathcal{G}(A)$ is an equivalence.*

Proof. Let $C = \pi_0 A \simeq \pi_0 B$. Proposition 3.4.5 implies that the result holds for the morphisms $A \to C$ and $B \to C$. It follows easily from this that the result holds for $A \to B$. □

Definition 3.4.7. A map $f : A \to B$ in $\mathcal{SCR}$ is *étale* (*smooth*) if it is formally étale (formally smooth) and almost of finite presentation.

Example 3.4.8. Let $A \in \mathcal{SCR}$, and let $a \in \pi_0 A$. Then $A[\frac{1}{a}]$ is an étale A-algebra. More precisely, take the free A-algebra $A[x]$, and let $A[\frac{1}{a}]$ be defined by attaching a 1-cell to kill $(xa - 1) \in \pi_0 A[x]$. One can then easily check that $\mathrm{Hom}_{\mathcal{SCR}}(A[\frac{1}{a}], B) \subseteq \mathrm{Hom}_{\mathcal{SCR}}(A, B)$ is the union of those connected components for which the induced map $\pi_0 A \to \pi_0 B$ carries a into an invertible element in $\pi_0 B$. Using this description it is easy to check that $A[\frac{1}{a}]$ is formally étale over A; since it is of finite presentation, it is étale over A.

Proposition 3.4.9. *Let $T : A \to A'$ be a morphism in $\mathcal{SCR}$. The following conditions are equivalent:*

1. *The morphism T is formally smooth, and A' is locally of finite presentation over A.*
2. *The morphism T is formally smooth, and A' is almost of finite presentation over A.*
3. *The morphism T is formally smooth, and $\pi_0 A'$ is a finitely presented algebra over $\pi_0 A$ in the category of ordinary commutative rings.*
4. *The morphism T is flat, and the induced morphism $\pi_0 A \to \pi_0 A'$ is a smooth homomorphism of ordinary commutative rings.*

Proof. It is clear that (1) implies (2) and that (2) implies (3). Suppose that (3) is satisfied. Considering small extensions of ordinary commutative rings, we deduce that $\pi_0 A \to \pi_0 A'$ is formally smooth in the usual sense so that $\pi_0 A'$ is a smooth $\pi_0 A$-algebra in the sense of ordinary commutative algebra. In particular, $\pi_0 A'$ is flat over $\pi_0 A$. Suppose first that $\pi_0 A' = (\pi_0 A)[x_1, \ldots, x_n]/(f_1, \ldots, f_k)$, where $k \times k$-minors of the Jacobian matrix of the relations $\{f_i\}$ generate the unit ideal of $\pi_0 A$. In this case, we use the same presentation to define an A-algebra $\widetilde{A}$. A simple calculation then shows that the cotangent complex $L_{\widetilde{A}/A}$ is projective, so that $\widetilde{A}$ is a smooth A-algebra. By Proposition 3.4.5, the isomorphism $\pi_0 \widetilde{A} \to \pi_0 A'$ lifts to a map $\widetilde{A} \to A'$. By construction, the natural map $f : L_{\widetilde{A}/A} \otimes_{\widetilde{A}} A' \to L_{A'/A}$ is a map of projective A'-modules which induces an isomorphism on π_0. It follows that f is an equivalence, so that $L_{A'/\widetilde{A}} = 0$. By Corollary 3.2.17, we deduce that $\widetilde{A} \to A'$ is an equivalence. Since $\widetilde{A}$ is flat over A, we get A' flat over A.

In the general case, we know that there exists a presentation for $\pi_0 A'$ having the above form *Zariski locally* on $\pi_0 A'$. The flatness of A' over A is equivalent to the assertion that certain maps $(\pi_n A) \otimes_{\pi_0 A} \pi_0 A' \to \pi_n A'$ be isomorphisms. This statement is local for the Zariski topology on $\pi_0 A'$; thus we deduce that (3) implies (4).

Now assume (4). The projectivity of $L_{A'/A}$ is local for the Zariski topology on $\pi_0 A'$, as is the property of being locally of finite presentation over A. Thus, we may assume that $\pi_0 A'$ admits a presentation as above. Lift this presentation to construct a flat A-algebra $\widetilde{A}$ and a map $g : \widetilde{A} \to A'$. Since g induces an isomorphism on π_0, the flatness of $\widetilde{A}$ and A' over A implies that g is an equivalence. Since $\widetilde{A}$ is formally smooth over A and finitely presented by construction, we deduce that A' is formally smooth over A. □

From Proposition 3.4.9 we can easily deduce the analogue for étale morphisms:

Corollary 3.4.10. *Let $T : A \to A'$ be a morphism in $\mathcal{SCR}$. The following conditions are equivalent:*

1. *The morphism T is formally étale and A' is of finite presentation as an A-algebra.*

2. *The morphism T is formally étale and A' is locally of finite presentation as an A-algebra.*

3. *The morphism T is formally étale and A' is almost of finite presentation over A.*

4. *The morphism T is formally étale, and $\pi_0 A'$ is a finitely presented algebra over $\pi_0 A$ in the category of ordinary commutative rings.*

5. *The morphism T is flat and the induced morphism $\pi_0 A \to \pi_0 A'$ is an étale homomorphism of ordinary commutative rings.*

Proof. The equivalence of (2), (3), (4), and (5) follows from Proposition 3.4.9. It is clear that (2) implies (1); the reverse implication follows from the vanishing of $L_{A'/A}$. □

Corollary 3.4.6 implies that if $f : A \to B$ is a morphism in $\mathcal{SCR}$ inducing an isomorphism $\pi_0 A \simeq \pi_0 B$, then the base change functor from étale A-algebras to étale B-algebras is fully faithful. We next study the essential surjectivity of this functor:

Proposition 3.4.11. *Let $f : A \to B$ be a morphism $\mathcal{SCR}$ which induces an isomorphism $\pi_0 A \simeq \pi_0 B$. Let B' be a smooth B-algebra. Then there exists a smooth A-algebra A' and an equivalence $B' \simeq B \otimes_A A'$.*

Proof. Suppose first that B is discrete. Then f is 1-connected, so that A may be obtained as the inverse limit of a tower of increasingly connected small extensions

$$\ldots \to A_2 \to A_1 \to A_0 = B.$$

It suffices to construct a compatible family $\{A'_n\}$ of smooth algebras over the family $\{A_n\}$; then we can construct A' as the inverse limit. We may therefore reduce to the case where A is a small extension of B by some B-module M. This small extension is classified by some $s \in \mathrm{Hom}_{\mathcal{M}_B}(L_{B/\mathbf{Z}}, M[1])$. Consider the exact triangle

$$L_{B/\mathbf{Z}} \otimes_B B' \to L_{B'/\mathbf{Z}} \to L_{B'/B}.$$

Since $L_{B'/B}$ is projective, we deduce that $\pi_{-1} \mathrm{Hom}_{\mathcal{M}_B}(L_{B'/B}, M[1]) = 0$, so that $s \otimes_B B'$ factors through some map $s' : L_{B'/\mathbf{Z}} \to M[1] \otimes_B B'$. This map classifies a small extension A' of B' by $M \otimes_B B'$. It is easy to see that A' is flat over A, hence smooth over A; moreover, the factorization of s through s' gives an identification of B' with $A' \otimes_A B$.

We now pass to the general case. Using the special case treated above, we see that $B' \otimes_B \pi_0 B$ can be lifted to a smooth A-algebra A'. Then $A' \otimes_A B$ is a smooth B-algebra which lifts $B' \otimes_B \pi_0 B$. It suffices to show that $A' \otimes_A B$ is equivalent to B'. Arguing inductively as above, it suffices to show that if $B \to C$ is a small extension of C by the C-module M, and B', B'' are smooth B-algebras, then any equivalence $B' \otimes_B C \to B'' \otimes_B C$ of C-algebras can be lifted to an equivalence of B-algebras. The obstruction to this lifting lies in $\pi_0 \operatorname{Hom}_{\mathcal{M}_{B'}}(L_{B'/B}, M[1] \otimes_B B'')$, which vanishes since $L_{B'/B}$ is projective. □

Remark 3.4.12. Proposition 3.4.11 is a generalization of the following classical fact: if X is a smooth affine algebraic variety, then any nth order deformation of X can be extended uniquely to an $(n+1)$st order deformation of X. This is because the obstruction to the existence and uniqueness of such an extension lie in $\mathrm{H}^2(X, T_X)$ and $\mathrm{H}^1(X, T_X)$, which vanish when X is affine.

For $A \in \mathcal{SCR}$, we let $\mathcal{SCR}^{\text{ét}}_{A/}$ denote the full subcategory of $\mathcal{SCR}_{A/}$ consisting of étale A-algebras. If B is an A-algebra, then $A' \mapsto A' \otimes_A B$ determines a functor $\mathcal{SCR}^{\text{ét}}_{A/} \to \mathcal{SCR}^{\text{ét}}_{A/}$. The following result plays a key philosophical role in the theory:

Theorem 3.4.13. *Let $\phi : A \to A'$ be a morphism in $\mathcal{SCR}$ which induces an isomorphism $\pi_0 A \simeq \pi_0 A'$. Then the base-change functor $\phi : \mathcal{SCR}^{\acute{e}t}_{A/} \to \mathcal{SCR}^{\acute{e}t}_{A'/}$ is an equivalence of ∞-categories.*

Proof. Corollary 3.4.6 implies that ϕ is fully faithful. The essential surjectivity follows from Proposition 3.4.11. □

Remark 3.4.14. We may interpret Theorem 3.4.13 as saying that the étale topos of a simplicial commutative ring A is identical with the étale topos of its ordinary ring of connected components $\pi_0 A$. This means that when we start gluing things together to make derived schemes, the gluing data are not really any more complicated than in classical algebraic geometry.

One should think of Theorem 3.4.13 as analogous to the classical assertion that the étale topology of a commutative ring A does not depend on the nilradical of A. Elements in the higher homotopy groups $\{\pi_i A\}_{i>0}$ may be thought of as "higher order nilpotent" elements of the structure sheaf of $\operatorname{Spec} A$. Like nilpotent elements, they have no classical interpretation as functions and do not affect the topology of $\operatorname{Spec} A$.

We conclude this section with a discussion of a weaker smoothness property:

Definition 3.4.15. A morphism $A \to B$ in $\mathcal{SCR}$ is *quasi-smooth* if B is almost of finite presentation over A and $L_{B/A}$ has Tor-amplitude ≤ 1.

Example 3.4.16. If k is a field, and A is a discrete k-algebra, then A is almost of finite presentation if and only if $\operatorname{Spec} A$ is a local complete intersection over $\operatorname{Spec} k$ in the classical sense.

We note that if B is quasi-smooth over A, then B is locally of finite presentation over A (since $L_{B/A}$ is almost perfect and of finite Tor-amplitude, and therefore perfect). Any smooth A-algebra is quasi-smooth. Moreover, if B and B' are quasi-smooth over A and admit maps $C \to B$, $C \to B'$ from some smooth A-algebra C, then $B \times_C B'$ is a quasi-smooth A-algebra (to prove this, just examine the cotangent complexes). Moreover, all quasi-smooth A-algebras arise in this way (at least locally). Indeed, if B is quasi-smooth over A, then there exists a surjection $C = A[x_1, \ldots, x_n] \to B$. Then $P = L_{B/A[x_1,\ldots,x_n]}[-1]$ is a projective B-module. Localizing C and B if necessary, we may suppose that P is free, and therefore B is obtained from C by killing a finite sequence of elements $\{y_1, \ldots, y_m\} \in \pi_0 C$. Now $B \simeq C \otimes_{A[y_1,\ldots,y_m]} A$.

The presentation given above shows that B is of finite Tor-amplitude over C (since A has Tor-amplitude $\leq m$ over $A[y_1, \ldots, y_m]$). Since C is flat over A, we may deduce the following:

Proposition 3.4.17. *If $A \to B$ is a quasi-smooth morphism in $\mathcal{SCR}$, then B is of finite* Tor*-amplitude as an A-module.*

The class of quasi-smooth morphisms is interesting because it seems to be the most general setting in which one has a good theory of virtual fundamental classes. This will be discussed in great detail in [23].

3.5 Properties of Modules and Algebras

At this point, we have introduced many properties for algebras and modules over a simplicial commutative ring (most of which generalize classical notions from commutative algebra). The goal of this section is summarize some of their interrelationships.

Definition 3.5.1. Let P be a property of modules over a simplicial commutative ring A. We shall say that P is *stable under arbitrary (étale, flat, smooth) base change* if whenever an A-module M has the property P and B is an arbitrary (étale, flat, smooth) A-algebra, then the B-module $B \otimes_A M$ has the property P.

We shall say that P is *local for the flat (étale, smooth) topology* if it has the following properties:

- P is stable under flat (étale, smooth) base change.
- Given a finite collection of objects $A_i \in \mathcal{SCR}$ and A_i-modules M_i, if each M_i has the property P as an A_i-module, then the product $\prod_i M_i$ has the property P as a $\prod_i A_i$-module.
- Whenever B is faithfully flat (étale and faithfully flat, smooth and faithfully flat) over A and M is an A-module such that the B-module $B \otimes_A M$ has the property P, then M also has the property P.

If P is a property of A-algebras, rather than A-modules, then the notions of stability under arbitrary (flat, étale, smooth) base change and locality for the flat (étale, smooth) topology are defined similarly.

The following proposition is easy and will henceforth be used without mention:

Proposition 3.5.2. • *The following properties of A-modules are stable under arbitrary base change: freeness, projectivity, flatness, faithful flatness, connectivity, being almost connective, being of* Tor*-amplitude* $\leq n$*, being of finite presentation, being perfect, being perfect to order* n*, being almost perfect, being zero.*

- *The following properties of A-algebras are stable under arbitrary base change: being of finite presentation, being locally of finite presentation, being of finite presentation to order* n*, being almost of finite presentation, flatness, faithful flatness, being étale, being formally étale, smoothness, formal smoothness, quasi-smoothness.*

The next proposition is little bit more difficult, but it is easily deduced from the corresponding flat descent theorems in classical algebraic geometry. See [11] for an extensive discussion.

Proposition 3.5.3. • *The following properties of A-modules are local for the flat topology: flatness, faithful flatness, being* n*-truncated, being connective, being almost connective, being of* Tor*-amplitude* $\leq n$*, being perfect, being perfect to order* n*, being almost perfect, being zero.*

- *The following properties of A-algebras are stable under flat descent: being locally of finite presentation, being of finite presentation to order* n*, being almost of finite presentation, being étale, being smoothness, being formally étale, formal smoothness, quasi-smoothness.*

Definition 3.5.4. Let P be a property of morphisms in $\mathcal{SCR}$. We say that P is *stable under composition* if it satisfies the following conditions:

- Any equivalence has the property P.
- If $A \to B$ and $B \to C$ are morphisms with the property P, then the composition $A \to C$ has the property P.
- If a morphism $f : A \to B$ has the property P, then any morphism homotopic to f also has the property P.

Proposition 3.5.5. *The following properties of morphisms are stable under composition: being of finite presentation, being locally of finite presentation, being almost finite presentation, being étale, being formally étale, smoothness, formal smoothness, flatness, faithful flatness, quasi-smoothness.*

Proposition 3.5.6. *Let* $A \in \mathcal{SCR}$*, and let* B *and* C *be A-algebras which are of finite presentation (locally of finite presentation, almost of finite presentation, étale, formally étale) over* A*. Let* f *be any A-algebra morphism from* B *to* C*. Then* C *is of finite presentation (locally of finite presentation, almost of finite presentation, étale, formally étale) over* B*.*

Definition 3.5.7. Let P be a property of morphisms of simplicial commutative rings. We shall say that P is *local on the source for the flat (étale, smooth) topology* if it satisfies the following conditions:

- Given any finite collection of morphisms $f_i : A \to B_i$ having the property P, the product morphism $A \to \Pi_i B_i$ has the property P.
- If $A \xrightarrow{f} B \xrightarrow{g} C$ is a composable pair of morphisms such that g is faithfully flat (étale and faithfully flat, smooth and faithfully flat), and $g \circ f$ has the property P, then f has the property P.

Proposition 3.5.8.

- *The following properties of morphisms are local on the source for the étale topology: étaleness, formal étaleness.*
- *The following properties of morphisms are local on the source for the smooth topology: being locally of finite presentation, being of finite presentation to order n, being almost of finite presentation, smoothness, formal smoothness, quasi-smoothness.*
- *The following properties of morphisms are local on the source for the flat topology: flatness, faithful flatness.*

We next justify some terminology which was introduced earlier by showing that in a precise sense, B is locally of finite presentation over A if and only if B can be "covered" by algebras which are of finite presentation over A:

Proposition 3.5.9. *Let $f : A \to B$ be a morphism in $\mathcal{SCR}$. The following are equivalent:*

1. *B is locally of finite presentation as an A-algebra.*
2. *There exists an étale B-algebra C which is faithfully flat over B and of finite presentation over A.*

Proof. If (1) is satisfied, then $L_{B/A}$ is perfect. Since every projective $\pi_0 B$-module becomes free Zariski-locally on $\pi_0 B$, there exists a faithfully flat, étale $\pi_0 B$ algebra $\pi_0 C$ such that $L_{C/A} \simeq L_{B/A} \otimes_B C$ is finitely presented, where C denotes the étale B-algebra lifting $\pi_0 C$. Then C is of finite presentation over B, hence locally of finite presentation over A; since $L_{C/A}$ is finitely presented, it is of finite presentation over A. This proves (2).

Assuming (2), we deduce that $\pi_0 B$ is a finitely presented $\pi_0 A$ algebra using classical descent arguments for ordinary commutative rings. Now is suffices to show that $L_{B/A}$ is perfect. By flat descent, it suffices to show that $L_{B/A} \otimes_B C \simeq L_{C/A}$ is perfect. But $L_{C/A}$ is finitely presented by assumption. This proves (1). □

3.6 Dualizing Modules

The purpose of this section is to describe the derived analogue of Grothendieck's theory of dualizing complexes. It turns out that Grothendieck duality theory can be adapted to derived algebraic geometry with very little additional effort, perhaps because the theory already has a bit of a derived flavor. We shall refrain from giving a complete exposition of this topic, since it will not be needed in this paper. However, we *will* need one component of this theory: the theory of dualizing modules in the "affine" case.

Definition 3.6.1. Let $A \in \mathcal{SCR}$ be Noetherian, and let K be an A-module. We shall say that K is a *dualizing module* if it has the following properties:

1. The module K is truncated and coherent.

2. The natural map of spectra $A \to \mathrm{Hom}_{\mathcal{M}_A}(K, K)$ is an equivalence.

3. The module K has *finite injective dimension.* That is, there exists $n \gg 0$ such that for any discrete A-module M, the A-module $\mathrm{Hom}_{\mathcal{M}_A}(M, K)$ is $(-n-1)$-connected. (In this case we shall say that K is of *injective dimension* $\leq n$.)

Remark 3.6.2. If A is a discrete commutative ring, then the notion of a dualizing module for A in the sense described above is equivalent to the notion of a *dualizing complex*: see [13].

Remark 3.6.3. If A is discrete and K is a dualizing module for A, then we may take $M = A$ in the third condition above, and thereby deduce that $\pi_i K = 0$ for $i \ll 0$. A similar argument may be applied if A is n-truncated for some n. In the general case where A has infinitely many nonvanishing homotopy groups, there is no reason to expect a dualizing module K to satisfy $\pi_i K = 0$ for $i \ll 0$. This is one feature of the derived duality theory which stands in sharp contrast to classical duality theory, and it leads to a few extra complications in the proofs given below.

Theorem 3.6.4. *Let $A \in \mathcal{SCR}$ be Noetherian, and suppose that K is a dualizing module. We define $M^\vee$ to be the A-module* $\mathrm{Hom}_{\mathcal{M}_A}(M, K)$.

1. *The functor $M \mapsto M^\vee$ induces a contravariant equivalence from the ∞-category of coherent A-modules to itself.*

2. *If M is coherent, then the natural map $M \to (M^\vee)^\vee$ is an equivalence.*

3. *Let M be a coherent A-module. Then M is almost perfect if and only if $M^\vee$ is truncated.*

Proof. We first show that if M is coherent, then $M^\vee$ is coherent. It suffices to show that each homotopy group $\pi_i M^\vee$ is finitely generated as a $\pi_0 A$-module. Since $M^\vee = \lim(\tau_{\geq -j} M)^\vee$, we deduce from the finite injective dimensionality of K that $\pi_i M^\vee = \pi_i (\tau_{\geq -j} M)^\vee$ for j large. Thus, we may replace M by $\tau_{\geq -j} M$ and suppose that M is almost perfect.

If we suppose that K is m-truncated, then $\pi_i M^\vee$ depends only on $\pi_j M$ for $j < m - i$. Thus, we may replace M by M', where $M' \to M$ is a highly connected map with M' finitely presented. We thereby reduce to the case where M is finitely presented. Using the appropriate exact triangles, we may reduce to the case where $M = A$. Then $M^\vee = K$ which is coherent by assumption.

To complete the proof of (1), it will suffice to prove (2) (since (2) exhibits the duality functor as its own homotopy inverse). Arguing as above, we note that $\pi_i(M^\vee)^\vee$ depends on only finitely many homotopy groups of $M^\vee$, which in turn depend on only finitely many homotopy groups of M. Thus we may again reduce to the case where M is perfect, and then to the case where $M = A$. Then $(M^\vee)^\vee = K^\vee \simeq A$, by the assumption that K is dualizing.

The proof of (3) is simple and left to the reader. □

As with the classical theory of dualizing complexes, it is easy to see that dualizing modules are in good supply:

Example 3.6.5. Let R be a (discrete) Gorenstein local ring. Then $K = R$ is a dualizing module for R.

Example 3.6.6. Let $A \in \mathcal{SCR}$ be Noetherian with a dualizing module, and let B be an A-algebra which is almost perfect as an A-module. Then $K' = \mathrm{Hom}_A(B, K)$ is a dualizing B-module. This follows easily from the adjunction formula $\mathrm{Hom}_{\mathcal{M}_B}(M, K') = \mathrm{Hom}_{\mathcal{M}_A}(M, K)$.

From Example 3.6.6, we deduce that if A has a dualizing module, then $\pi_0 A$ has a dualizing module. In more classical language, this dualizing module is a *dualizing complex* in the sense of Grothendieck. The existence of such a dualizing complex implies that $\pi_0 A$ has finite Krull dimension (see [13]). We shall say that A is *of finite Krull dimension* if $\pi_0 A$ is of finite Krull dimension, so that any Noetherian $A \in \mathcal{SCR}$ having a dualizing module is of finite Krull dimension.

Lemma 3.6.7. *Let $A \in \mathcal{SCR}$ be Noetherian and let K be a truncated A-module. Then K is of injective dimension $\leq n$ if and only if $\pi_i \mathrm{Hom}_{\mathcal{M}_A}(M, K)$ for each $i < -n$ and each finitely generated discrete A-module M.*

Proof. Replacing K by $K[n]$, we may suppose that $n = 0$. Replacing A by $\pi_0 A$ and K by $\mathrm{Hom}_{\mathcal{M}_A}(\pi_0 A, K)$, we may suppose that A is discrete. Since K is truncated, we may represent K by a complex

$$0 \to I_m \to I_{m-1} \to \dots$$

of discrete, injective A-modules. Let I denote the A-module represented by the complex

$$0 \to I_m \to I_{m-1} \to \dots \to I_1 \to 0 \to \dots.$$

Then I is of injective dimension ≤ -1 and there is a triangle $I' \to K \to I$, where I' is 0-truncated. To complete the proof, it suffices to show that I' is discrete and is an injective object in the abelian category of discrete A-modules.

For any discrete A-module M, we obtain a long exact sequence

$$\ldots \to \pi_i \operatorname{Hom}_{\mathcal{M}_A}(M, I') \to \pi_i \operatorname{Hom}_{\mathcal{M}_A}(M, K) \to \pi_i \operatorname{Hom}_{\mathcal{M}_A}(M, I) \to \pi_{i-1} \operatorname{Hom}_{\mathcal{M}_A}(M, I') \to \ldots$$

This exact sequence implies that $\pi_i \operatorname{Hom}_{\mathcal{M}_A}(M, I') = 0$ for $i < 0$ when M is finitely generated. In particular, taking $M = A$, we deduce that I' is discrete.

If

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence of finitely generated discrete A-modules, then the vanishing of $\pi_{-1} \operatorname{Hom}_{\mathcal{M}_A}(M'', I')$ implies that the induced sequence

$$0 \to \pi_0 \operatorname{Hom}_{\mathcal{M}_A}(M'', I') \to \pi_0 \operatorname{Hom}_{\mathcal{M}_A}(M, I') \to \pi_0 \operatorname{Hom}_{\mathcal{M}_A}(M', I') \to 0$$

is exact. Since A is Noetherian, it follows that the induced map $I' \to \operatorname{Hom}_{\mathcal{M}_A}(J, I')$ is surjective for any ideal $J \subseteq A$. Using Zorn's lemma we may deduce that I' is injective. □

Theorem 3.6.8. *Let $A \in \mathcal{SCR}$ be Noetherian. Suppose that A has a dualizing module. If B is any A-algebra which is almost of finite presentation, then B also has a dualizing module.*

Proof. We may realize B as an almost perfect $A[x_1, \ldots, x_n]$ module for some map $A[x_1, \ldots, x_n] \to B$. Using Example 3.6.6, we may reduce to the case where $B = A[x_1, \ldots, x_n]$. Working by induction on n, we may reduce to the case where $B = A[x]$.

Let K be a dualizing module for A. We claim that $K[x] = K \otimes_A A[x]$ is a dualizing module for $A[x]$. It is clear that $K[x]$ is truncated and coherent. We next claim that the formation of $K[x]$ is compatible with "finite" base change in A. Namely, suppose that $A \to A'$ expresses A' as an almost perfect A-module. Then $K' = \operatorname{Hom}_{\mathcal{M}_A}(A', K)$ is a dualizing module for A'. We claim that the natural map $K'[x] \to \operatorname{Hom}_{A[x]}(A'[x], K[x])$ is an equivalence. We may rewrite the target as $\operatorname{Hom}_{\mathcal{M}_A}(A', K[x]) = \operatorname{Hom}_{\mathcal{M}_A}(A', \oplus_i K x^i)$. Since A' is almost perfect as an A-module, $\operatorname{Hom}_{\mathcal{M}_A}(A', \bullet)$ commutes with infinite direct sums when restricted to truncated modules. Since K is truncated, the claim follows.

We now prove that $K[x]$ has finite injective dimension as an $A[x]$-module. Suppose that K is of injective dimension $\leq n$. The existence of K implies that A has finite Krull dimension. It follows that $A[x]$ has Krull dimension $\leq m$ for some m. We will show that $K[x]$ has injective dimension $\leq n + m + 1$.

It will suffice to show that if M is a finitely generated, discrete $A[x]$-module, then $\pi_i \operatorname{Hom}_{\mathcal{M}_{A[x]}}(M, K[x]) = 0$ for $i < -n - m - 1$. We will show, more generally, that if the support of M has Krull dimension $\leq j$, then $\pi_i \operatorname{Hom}_{\mathcal{M}_{A[x]}}(M, K[x]) = 0$ for $i < -n - j - 1$. We prove this by induction on j. Filtering M and working by induction, we may suppose that $M \simeq (\pi_0 A)[x]/\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of $(\pi_0 A)[x]$. Let $\mathfrak{q} = \mathfrak{p} \cap \pi_0 A$. Replacing A by $\pi_0 A/\mathfrak{q}$, we may reduce to the case where A is a discrete integral domain and $\mathfrak{q} = 0$. If $\mathfrak{p} = 0$, then $\operatorname{Hom}_{\mathcal{M}_{A[x]}}(M, K[x]) \simeq K[x] = \operatorname{Hom}_{\mathcal{M}_A}(A, K)[x]$, whose homotopy groups vanish in degrees $< -n$ by assumption.

If $\mathfrak{p} \neq 0$, then we may choose $y \in \mathfrak{p}$ which generates $\mathfrak{p}$ after tensoring with fraction field of A. Then there is an exact sequence

$$0 \to N \to A[x]/(y) \to M \to 0.$$

To show that $\pi_i \operatorname{Hom}_{\mathcal{M}_{A[x]}}(M, K[x]) = 0$, it suffices to show that $\pi_i \operatorname{Hom}_{\mathcal{M}_{A[x]}}(A[x]/(y), K[x]) = 0 = \pi_{i+1} \operatorname{Hom}_{\mathcal{M}_{A[x]}}(N, K[x]))$. Since the support of N has Krull dimension strictly less than j, the vanishing of $\pi_{i+1} \operatorname{Hom}_{\mathcal{M}_{A[x]}}(N, K[x]))$ follows from the inductive hypothesis. Using the exact sequence $0 \to A[x] \to A[x] \to A[x]/(y) \to 0$, the vanishing of $\pi_i \operatorname{Hom}_{\mathcal{M}_{A[x]}}(A[x]/(y), K[x])$ can be deduced from the vanishing of $\pi_i K[x]$ and $\pi_{i+1} K[x]$. Since $i+1 < -n$, this follows from the assumption that K has injective dimension $\leq n$ (since we have reduced to the case where A is discrete).

It remains to show that $K[x]$ satisfies biduality. We have $\operatorname{Hom}_{\mathcal{M}_{A[x]}}(K[x], K[x]) = \operatorname{Hom}_{\mathcal{M}_A}(K, K[x])$. We must show that the natural map $A[x] \simeq \operatorname{Hom}_{\mathcal{M}_A}(K, K)[x] \to \operatorname{Hom}_{\mathcal{M}_A}(K, K[x])$ is an equivalence. It suffices to check this map induces an isomorphism on each homotopy group π_i. Since $K[x]$ has finite injective dimension, we have $\pi_i \operatorname{Hom}_{\mathcal{M}_{A[x]}}(K[x], K[x]) \simeq \pi_i \operatorname{Hom}_{\mathcal{M}_{A[x]}}(\tau_{\geq k} K[x], K[x])$ for k sufficiently small. Similarly, $\pi_i \operatorname{Hom}_{\mathcal{M}_A}(K, K) \simeq \pi_i \operatorname{Hom}_{\mathcal{M}_A}(\tau_{\geq k} K, K)$, so that it suffices to show that the natural map $\operatorname{Hom}_{\mathcal{M}_A}(\tau_{\geq k} K, K)[x] \to \operatorname{Hom}_{\mathcal{M}_A}(\tau_{\geq k} K, K[x])$ is an equivalence. This follows immediately from the assumption that $K[x]$ is truncated and $\tau_{\geq k} K$ is almost perfect. □

We are now prepared to prove the main result of this section. We remark that the conclusion of the result does not mention dualizing modules: these instead enter as a tool in the proof. It seems likely that a more direct proof is possible, which would enable one to eliminate the hypothesis that A admit a dualizing module. However, we were unable to find such a proof.

Theorem 3.6.9. *Let $A \in \mathcal{SCR}$ be Noetherian, and suppose that A has a dualizing module K. Let $\mathcal{F} : \mathcal{M}_A \to \mathcal{S}_\infty$ be an exact functor. Then there exists an almost perfect A-module M and an identification of $\mathcal{F}$ with the functor* $\operatorname{Hom}_{\mathcal{M}_A}(M, \bullet)$ *if and only if the following conditions are satisfied:*

1. *For each $N \in \mathcal{M}_A$, we have $\mathcal{F}(N) = \lim\{\mathcal{F}(\tau_{\leq n} N)\}$.*

2. *The functor $\mathcal{F}$ commutes with filtered colimits when restricted to $(\mathcal{M}_A)_{\leq 0}$.*

3. *For every finitely generated (discrete) $\pi_0 A$-module N, the functor $\pi_0 \mathcal{F}(N[-i])$ is a finitely generated module.*

4. *There exists an integer n such that $\pi_i \mathcal{F}(N) = 0$ for all $N \in (\mathcal{M}_A)_{\leq 0}$ and all $i \geq n$.*

Proof. It is obvious that all four conditions are necessary. Conversely, suppose that they are each fulfilled. For each $N \in \mathcal{M}_A$, the A-module structure on N naturally induces an A-module structure on $\mathcal{F}(N)$, so that we may view $\mathcal{F}$ as A-module valued.

Conditions (3) and (4) imply that if N is discrete and finitely generated as a $\pi_0 A$-module, then $\mathcal{F}(N)$ is truncated and coherent. Using condition (4) and induction, we deduce that $\mathcal{F}(N)$ is truncated and coherent whenever N is truncated and coherent.

Let K be a dualizing module for A, and let $N \mapsto N^\vee$ be the associated duality functor. Now set $\mathcal{G}(N) = \mathcal{F}(N^\vee)^\vee$. We note that $\mathcal{G}$ carries almost perfect A-modules to almost perfect A-modules. Using the evident A-linearity, one may construct a natural transformation $\phi_N : N \otimes \mathcal{G}(A) \rightarrow \mathcal{G}(N)$.

Our next goal is to prove that ϕ_N is an equivalence whenever N is almost perfect. This will be achieved in several steps:

- The map $\phi_{A[n]}$ is an equivalence for any $n \in \mathbf{Z}$. This is immediate from the definition.
- If $N' \rightarrow N \rightarrow N''$ is an exact triangle of almost perfect A-modules, and both $\phi_{N'}$ and $\phi_{N''}$ are equivalences, then ϕ_N is an equivalence.
- The map ϕ_N is an equivalence whenever N is finitely presented. This follows by induction, using the last two steps.
- Since K is a dualizing module, any fixed homotopy group of $N^\vee$ depends on only finitely many homotopy groups of N. Similarly, condition (4) implies that any fixed homotopy group of $\mathcal{F}(N)$ is unchanged by replacing N by $\tau_{\geq n} N$ for n sufficiently negative. Putting these facts together, we deduce that $\pi_i \, \mathcal{G}(N) \simeq \pi_i \, \mathcal{G}(\tau_{\leq n+i} N)$ for some $n \gg 0$.
- To prove that ϕ_N is an equivalence, it suffices to prove that ϕ_N induces an equivalence on homotopy groups. For sufficiently large n, both $\pi_i \, \mathcal{G}(N)$ and $\pi_i(\mathcal{G}(A) \otimes_A N)$ depend only on $\tau_{\leq i+n} N$. Thus, if N is almost perfect, then we are free to replace N by a finitely presented A-module N' which closely approximates N in the sense that $N' \rightarrow N$ is $(i+n)$-connected. We thereby reduce to the case where N is finitely presented which was handled above.

We now set $M = \mathcal{G}(A)$. If N is truncated and coherent, then we have natural equivalences

$$\mathrm{Hom}_{\mathcal{M}_A}(M, N) \simeq \mathrm{Hom}_{\mathcal{M}_A}(M, \mathrm{Hom}_{\mathcal{M}_A}(N^\vee, K)) \simeq \mathrm{Hom}_{\mathcal{M}A}(M \otimes_A N^\vee, K) \simeq \mathcal{G}(N^\vee)^\vee = \mathcal{F}(N)$$

as A-modules. In other words, the functors $\mathrm{Hom}_{\mathcal{M}_A}(M, \bullet)$ and $\mathcal{F}$ are equivalent when restricted to k-truncated, coherent modules (for any $k \in \mathbf{Z}$). Since both functors are compatible with filtered colimits, they have equivalent restrictions to the ∞-category of all k-truncated A-modules. Finally, since both $\mathrm{Hom}_{\mathcal{M}_A}(M, \bullet)$ and $\mathcal{F}$ satisfy condition (1) of the proposition, we deduce that they are equivalent on all of $\mathcal{M}_A$. □

3.7 Popescu's Theorem

In this section we discuss Popescu's theorem, which asserts that a geometrically regular morphism of Noetherian rings may be approximated by smooth ring homomorphisms. We will begin by reviewing the requisite commutative algebra.

We first recall that many useful properties of Noetherian rings may be generalized to *relative* properties, using the following prescription: a morphism $f : A \to B$ is said to have the property P if f is flat, and if the ring $B \otimes_A \kappa$ has the property P whenever κ is a residue field of A. In these cases, we can use this same relative definition in derived commutative algebra, since the flatness of f implies that $B \otimes_A \kappa$ is a discrete κ-algebra.

We will apply this idea to the particular case of *geometrically regular* morphisms of Noetherian rings. Recall that a morphism $A \to B$ of Noetherian rings is said to be *geometrically regular* if it is flat and if $B \otimes_A \kappa$ is a regular Noetherian ring whenever κ is a finite extension of some residue field of A. We will take this as our definition of geometrically regular morphisms of simplicial commutative rings (assumed again to be Noetherian). Equivalently, a morphism $A \to B$ is between Noetherian objects of $\mathcal{SCR}$ is *geometrically regular* if B is flat over A and $\pi_0 B$ is a geometrically regular $\pi_0 A$-algebra.

We recall the classical version of Popescu's theorem:

Theorem 3.7.1. *Let $f : A \to B$ be a map of (ordinary) Noetherian rings. The map f is geometrically regular if and only if B can be written as a filtered colimit of smooth A-algebras.*

For a proof, we refer the reader to [33]. We will use this theorem to deduce a version of Popescu's theorem for simplicial commutative rings. First, we need several lemmas:

Lemma 3.7.2. *Let R be a local Noetherian (discrete) ring with residue field k. Let $\{x_1, \ldots, x_n\}$ be a system of generators for the maximal ideal $\mathfrak{m}$ of R, whose images in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent. If $\mathrm{Tor}_1^{\mathbf{Z}[x_1,\ldots,x_n]}(R, \mathbf{Z}) = 0$, then R is regular.*

Proof. First, suppose that R is the quotient of a regular Noetherian local ring $\widetilde{R}$. Replacing $\widetilde{R}$ by a quotient if necessary, we may assume that the embedding dimensions of $\widetilde{R}$ and R are the same. Then we can lift the sequence $\{x_i\}$ to a regular system of parameters $\{\widetilde{x}_i\}$ in R. This choice gives a factorization of $\mathbf{Z}[x_1, \ldots, x_n] \to R$ through $\widetilde{R}$. Making use of a change-of-rings isomorphism, we see that the group $\mathrm{Tor}_1^{\mathbf{Z}[x_1,\ldots,x_n]}(R, \mathbf{Z})$ is isomorphic to $\mathrm{Tor}_1^{\widetilde{R}}(R, k)$. Since $\widetilde{R}$ is local and Noetherian and R is a finite module over $\widetilde{R}$, we deduce that R is a flat $\widetilde{R}$-module. In particular, it is torsion-free as a $\widetilde{R}$-module, so that $R \simeq \widetilde{R}$ and R is regular.

In the general case, we note that the group $\mathrm{Tor}_1^{\mathbf{Z}[x_1,\ldots,x_n]}(R, \mathbf{Z})$ is a finite R-module whose formation is compatible with flat base change in R. Since the completion of R is always the quotient of a regular local ring, we deduce that the completion of R is regular, so that R is regular. □

For the statement of the next lemma, we introduce a bit of terminology. An object $R \in \mathcal{SCR}$ will be said to be *local* if $\pi_0 R$ is local. In this case, we define the *residue field* of R to be the residue field of $\pi_0 R$.

Lemma 3.7.3. *Let $R \in \mathcal{SCR}$ be local and Noetherian with residue field K. Then the following conditions are equivalent:*

1. *The ring R is discrete and regular.*

2. *The homotopy groups $\pi_i L_{K/R}$ vanish for $i \neq 1$.*

Proof. By Proposition 3.2.16, we have a natural isomorphism $\pi_1 L_{K/R}$ with $\mathfrak{m}/\mathfrak{m}^2$, where $\mathfrak{m}$ denotes the maximal ideal in $\pi_0 R$.

Choose a minimal set of generators $\{x_1, \ldots, x_n\}$ for $\mathfrak{m}$. Let $\widetilde{K} = R \otimes_{\mathbf{Z}[x_1,\ldots,x_n]} \mathbf{Z}$. Then we have a natural map $\widetilde{K} \rightarrow K$ which induces an isomorphism on π_0. By construction, $L_{\widetilde{K}/R}$ is freely generated by elements of degree 1 corresponding to the x_i, so that $f : L_{\widetilde{K}/R} \otimes_{\widetilde{K}} K \rightarrow L_{K/R}$ induces an isomorphism on π_1. Thus, (2) is equivalent to the assertion that f is an equivalence. This is in turn equivalent to the assertion that $L_{\widetilde{K}/K} = 0$. Since $K = \pi_0 \widetilde{K}$, we see that (2) is equivalent to the assertion that $\widetilde{K} = K$. It is now clear that (1) implies (2).

We now use the appropriate spectral sequence to compute the homotopy groups of $\widetilde{K}$. This spectral sequence has E_2^{pq}-term given by $\mathrm{Tor}_p^{\mathbf{Z}[x_1,\ldots,x_n]}(\pi_q R, \mathbf{Z})$. In particular, we have an exact sequence of low degree terms

$$\mathrm{Tor}_0^{\mathbf{Z}[x_1,\ldots,x_n]}(\pi_1 R, \mathbf{Z}) \rightarrow \pi_1 \widetilde{K} \rightarrow \mathrm{Tor}_1^{\mathbf{Z}[x_1,\ldots,x_n]}(\pi_0 R, \mathbf{Z}) \rightarrow 0.$$

Now, if (2) holds, then $\pi_1 \widetilde{K} = 0$. The exact sequence shows that $\mathrm{Tor}_1^{\mathbf{Z}[x_1,\ldots,x_n]}(\pi_0 R, \mathbf{Z}) = 0$. By Lemma 3.7.2, this implies that $\pi_0 R$ is regular.

Suppose R is not discrete. Choose m minimal such that $\pi_m R \neq 0$. Since $\pi_m R$ is a finite module over $\pi_0 R$, we see that E_2^{0m} is nonzero. Since $\pi_m \widetilde{K} = 0$, we see that some differential $E_k^{r(m+1-r)} \rightarrow E_r^{0m}$ must be nonzero, $r \geq 2$. This implies $E_2^{r(m+1-r)} \neq 0$. By the minimality of m, this is impossible unless $r = m+1$, in which case we get $E_2^{r(m+1-r)} = \mathrm{Tor}_{m+1}^{\mathbf{Z}[x_1,\ldots,x_n]}(R, \mathbf{Z}) = 0$ from the regularity of R. $\square$

Lemma 3.7.4. *Let $A \in \mathcal{SCR}$ be Noetherian, and let M be a connective A-module. Then M is flat if and only if $M \otimes_A \kappa$ is discrete for any residue field κ of A.*

Proof. The "only if" direction is clear. For the "if", let us suppose that M is *not* flat. Then there exists a discrete A-module N such that $M \otimes_A N$ is not discrete. Since tensor products commute with filtered colimits, we may assume that N is finitely presented when regarded as a $\pi_0 A$-module in the usual sense. Since $\pi_0 A$ is Noetherian, we may assume N to be chosen so that its annihilator ideal $I \subseteq \pi_0 A$ is as large as possible. Replacing A by $\pi_0 A/I$ and M by $M \otimes_A (\pi_0 A/I)$, we may assume that A is discrete and that N is a faithful A-module.

We first claim that A is an integral domain. Indeed, suppose that $xy = 0$ in A. Let $N' = \{n \in N : xn = 0\}$ and let $N'' = N/N'$. Then a long exact sequence shows that either $M \otimes_A N'$ or $M \otimes_A N''$ is nondiscrete. By maximality, this implies that either $x = 0$ or $y = 0$.

Let N_0 denote the torsion submodule of N. Since N_0 has a larger annihilator than N, $N_0 \otimes_A M$ is discrete. Consequently, a long exact sequence shows that $(N/N_0) \otimes_A M$ must be nondiscrete. Replacing N by N/N_0, we may suppose that N is torsion-free.

For any nonzero element $x \in A$, we have a short exact sequence

$$0 \to N \xrightarrow{x} N \to N/xN \to 0.$$

Since $(N/xN) \otimes_A M$ is discrete, we deduce that multiplication by x induces an isomorphism on $\pi_n(N \otimes_A M)$ for $n > 0$. Since this holds for all $x \in A$, it follows that $\pi_n(N \otimes_A M) \simeq \pi_n((N \otimes_A \kappa) \otimes_A M)$, where κ denotes the field of fractions of A. Replacing N by $N \otimes_A \kappa$, we may suppose that N is a κ-vector space. Then N is a direct sum of copies of κ. It follows that $\kappa \otimes_A M$ is nondiscrete, and the proof is complete. □

Theorem 3.7.5 (Derived Popescu Theorem). *Let $f : A \to B$ be a morphism in $\mathcal{SCR}$. Assume that A and B are Noetherian. The following conditions are equivalent:*

1. *For any factorization $A \to C \xrightarrow{g} B$, where C is locally of finite presentation as an A-algebra, there exists a factorization $C \to D \to B$ of g such that D is smooth over A.*

2. *The A-algebra B is a filtered colimit of smooth A-algebras.*

3. *The cotangent complex $L_{B/A}$ is a flat B-module.*

4. *The morphism f is geometrically regular.*

Proof. The ∞-category $\mathcal{SCR}_{A//B}$ is compactly generated. Since it has a final object, its compact objects form a filtered ∞-category $\mathcal{SCR}^c_{A//B}$; moreover, these algebras have B as their filtered colimit. Condition (1) ensures that the full subcategory of $\mathcal{SCR}^c_{A//B}$ consisting of smooth A-algebras is cofinal. Consequently, this ∞-category is also filtered and it has the same filtered colimit B. Thus, we see that (1) implies (2). The implication (2) implies (3) is clear because the formation of the cotangent complex is compatible with filtered colimits, and a filtered colimit of projective modules is flat.

We show that (3) implies (4). Using Lemma 3.7.4, we can reduce to the case where A is a field k. In this case, we need to show that the flatness of $L_{B/k}$ implies that B is discrete and regular. Replacing B by one of its localizations, we may assume that $\pi_0 B$ is local with residue field K. Now consider the triangle

$$L_{B/k} \otimes_B K \to L_{K/k} \to L_{K/B}.$$

Since $L_{B/k}$ is flat and $L_{K/k}$ is 1-truncated, we deduce that $L_{K/B}$ is 1-truncated. The surjectivity of the map $B \to K$ then shows that $\pi_i L_{K/B} = 0$ for $i \neq 1$. By Lemma 3.7.3, we deduce that B is discrete and regular.

Now suppose that (4) is satisfied. We will deduce (1) using the same argument that we used in the proof of Theorem 2.5.2. First, since any A-algebra C which is locally of finite presentation is the retract of a finitely presented A-algebra, we may reduce to the case where C is finitely presented. In this case, there exists a finite sequence of A-algebras $A = C_0 \to \ldots \to C_n = C$, where each C_i is obtained from C_{i-1} by attaching a k-cell for some $k \geq 0$. We will prove, by induction on i, that there exists a C_i-algebra D_i which is smooth over A, and a factorization of the map $C_i \to B$ through D_i. For $i = 0$, we simply take $D_0 = A$.

For the inductive step, let us suppose that D_i has already been constructed. We must show that it is possible to construct D_{i+1}. Replacing C_i by D_i and C_{i+1} by $C_{i+1} \otimes_{C_i} D_i$, we may assume that C_{i+1} is obtained from D_i by adjoining a k-cell for some $k \geq 0$. If $k = 0$, then $C_{i+1} = D_i[x]$ is smooth over A and we may take $D_{i+1} = C_{i+1}$.

Suppose next that $k > 0$, and let C_{i+1} be obtained from C_i by attaching a cell to kill $x \in \pi_{k-1} C_i = \mathrm{Tor}_0^{\pi_0 A}(\pi_0 C_i, \pi_{k-1} A)$. Then the image of $x \in \pi_{k-1} B = \mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_k A)$ vanishes. By Theorem 3.7.1, $\pi_0 B$ is a filtered colimit of smooth $\pi_0 A$ algebras, so that there exists a factorization $\pi_0 C_i \to \pi_0 D' \to B$, where $\pi_0 D'$ is smooth over $\pi_0 A$ and the image of x vanishes in $\mathrm{Tor}_0^{\pi_0 A}(\pi_0 D', \pi_{k-1} A)$. By Proposition 3.4.11, we can lift $\pi_0 D'$ to a smooth A-algebra D. Using Proposition 3.4.5, the maps $f_0 : \pi_0 C_i \to \pi_0 D'$ and $g_0 : \pi_0 D' \to \pi_0 B$ can be lifted to maps $f : C_i \to D'$ and $g : D' \to B$. Unfortunately, it is not necessarily that case that $g \circ f$ is homotopic to h. Indeed, there is an obstruction $\xi \in \pi_1 \mathrm{Hom}_{\mathcal{M}_B}(L_{C_i/A}, B)$. Let us regard g as fixed, and try to kill the obstruction by varying f. Since f is ambiguous up to the group $\pi_1 \mathrm{Hom}_{\mathcal{M}_{C_i}}(L_{C_i/A}, D')$, we see that an appropriate choice of f exists provided that ξ lies in the image of $\pi_1 \mathrm{Hom}_{\mathcal{M}_{C_i}}(L_{C_i/A}, D') \to \pi_1 \mathrm{Hom}_{\mathcal{M}_{C_i}}(L_{C_i/A}, B)$. Replacing D' by a free algebra $D'[y_1, \ldots, y_m]$, we may ensure that the image of $\pi_1 D' \to \pi_1 B$ is arbitrarily large, so that the required factorization can be found. Replacing C_i by D' and C_{i+1} by $C_{i+1} \otimes_{C_i} D'$, we may reduce to the case where $x = 0$.

Now C_{i+1} is the free C_i-algebra on a k-cell, having image $y \in \pi_k B$. Repeating the above argument, we may find a factorization $C_i \to D'' \to B$ with the property that y lies in the image of $\pi_k D'' \to \pi_k B$. It follows immediately that $C_{i+1} \to B$ factors through D'', as desired. □

Remark 3.7.6. The equivalence (3) ⇔ (4), for ordinary commutative rings, is proven in [1]. We note that conditions (1), (2), and (3) can be formulated in the absence of any Noetherian hypotheses on the rings A and B. In the non-Noetherian setting, it is easy to see that (1) ⇔ (2) ⇒ (3). It seems reasonable to conjecture that (3) ⇒ (1), at least when B is flat over A. The proof given above shows that if B is flat over A, the implication (3) ⇒ (1) follows in general once it is known for ordinary commutative rings.

Popescu's theorem is frequently useful in the following situation. Let $A \in \mathcal{SCR}$ be Noetherian and local (meaning that $\pi_0 A$ is Noetherian and local and each $\pi_i A$ is a finite $\pi_0 A$-module). In §6, we shall define a completion $\hat{A}$, which will have the property that $\pi_i \hat{A}$ is the completion of $\pi_i A$ with respect to the $\mathfrak{m}$-adic topology, where $\mathfrak{m} \subseteq \pi_0 A$ denotes the maximal ideal. Then $\hat{A}$ is a flat A-algebra. Under reasonable circumstances (for example, whenever $\pi_0 A$ is excellent), the morphism $A \to \hat{A}$ is geometrically regular. Theorem 3.7.5 implies that $\hat{A}$ is a filtered colimit of smooth A-algebras. This gives a strong form of the Artin approximation theorem, which can be used to simplify the proof of Artin's representability theorem: see [8]. Our proof will make use of the same strategy, together with some additional simplifications which become available in the derived setting.

Chapter 4

Derived Schemes

In this section, we explain how to use the "derived commutative algebra" developed in §3 to define derived schemes. We will begin in §4.1 with a discussion of sheaves on ∞-topoi with values in an ∞-category $\mathcal{C}$, and discuss an appropriate theory of classifying ∞-topoi. In §4.2, we will specialize to the case where $\mathcal{C} = \mathcal{SCR}$ and consider various topologies on commutative rings along with their generalizations to the case of simplicial commutative rings.

In §4.3 we will discuss the construction of "spectra" (in the sense of commutative algebra) based on these topologies. Our approach to this question is somewhat less direct than is usual: we first define the spectrum of a ring (or, more generally, a $\mathcal{SCR}$-valued sheaf) by a certain universal property. We then prove the existence of an object having this universal property by a somewhat familiar-looking explicit construction. Our setup is very general, and the ideas could conceivably be useful for studying algebraic structures other than commutative rings. It is also well-adapted to relative situations (in the case of Zariski spectra of commutative rings, it recovers the relative spectrum construction discussed in [12]).

The essentially combinatorial origin of the spectrum construction implies that the underlying (∞)-topoi of derived schemes have good finiteness properties, which we spell out in §4.4. Finally, in §4.5 we give the definition of a derived scheme, and compare derived schemes with classical algebro-geometric notions such as schemes, algebraic spaces, and Deligne-Mumford stacks.

The purpose of this section is to give a definition of derived schemes which is analogous to the original definition of a scheme: it is something like a space, equipped with a sheaf of rings, which locally takes a particularly simple form. In some more abstract approaches to the theory, one views a scheme as a certain kind of set-valued functor on the category of commutative rings. In §4.6 we show that a derived scheme X is determined by the $\mathcal{S}$-valued functor $A \mapsto \mathrm{Hom}(\mathrm{Spec}\, A, X)$ on $\mathcal{SCR}$, so that it is also possible to give a purely functorial approach to derived algebraic geometry. This paves the way for our discussion of derived stacks in §5. In fact, our development of the theory of derived stacks is for the most part independent of the material of the present section, so the reader can skip ahead to §5 with little loss of continuity.

4.1 Structure Sheaves and Classifying ∞-Topoi

We begin our discussion with some elementary definitions concerning ∞-topoi equipped with "structure sheaves". For the time being, these may take values in any ∞-category $\mathcal{C}$, though the main case of interest to us will be when $\mathcal{C} = \mathcal{SCR}$.

Definition 4.1.1. Let $\mathcal{X}$ be an ∞-topos, and $\mathcal{C}$ any ∞-category. A *$\mathcal{C}$-valued presheaf* on $\mathcal{X}$ is a contravariant functor $\mathcal{X} \to \mathcal{C}$. A *$\mathcal{C}$-valued sheaf* on $\mathcal{X}$ is a $\mathcal{C}$-valued presheaf which carries colimits into limits. If $f : \mathcal{X} \to \mathcal{Y}$ is a geometric morphism of ∞-topoi, and $\mathcal{O}$ is a $\mathcal{C}$-valued (pre)sheaf on $\mathcal{X}$, then $f_* \mathcal{O}$ is the $\mathcal{C}$-valued sheaf on $\mathcal{Y}$ obtained by composing the pullback functor f^* with the functor $\mathcal{O}$.

Remark 4.1.2. One advantage of working with ∞-topoi is that Definition 4.1.1 becomes very simple. If we instead worked with ordinary topoi (or n-topoi), then the above definition would be correct only if $\mathcal{C}$ is itself an ordinary category (or an n-category). Thus, even though we are primarily interested in ∞-topoi which are associated to ordinary topoi, there is some value in regarding them as ∞-topoi if we wish to discuss sheaves with values in an ∞-category.

Remark 4.1.3. We have chosen to call a limit-preserving functor $\mathcal{O} : \mathcal{X}^{op} \to \mathcal{C}$ a *$\mathcal{C}$-valued sheaf* on $\mathcal{X}$. One might just as well refer to such a functor as a *$\mathcal{C}$-valued object* of $\mathcal{X}$. For example, if $\mathcal{C}$ is the (ordinary) category of abelian groups, then one may identify $\mathcal{C}$-valued sheaves on $\mathcal{X}$ with the category of abelian group objects in the ordinary category $\tau_{\leq 0} \mathcal{X}$ of discrete objects in $\mathcal{X}$.

The former terminology seems more in line with the point of view that an ∞-topos is some kind of generalized topological space, while the latter emphasizes the role of $\mathcal{X}$ as a "place where one can do mathematics". Both points of view are valuable, but we feel that the first is more in line with the objective of this paper.

Example 4.1.4. Let $\mathcal{C} = \mathcal{S}$ be the ∞-category of spaces. Then the $\mathcal{C}$-valued sheaves on $\mathcal{X}$ are precisely those presheaves of spaces on $\mathcal{X}$ which transform colimits into limits; in other words, they are precisely the representable presheaves on $\mathcal{X}$. Thus, $\mathcal{X}$ may be identified with the ∞-category of $\mathcal{S}$-valued sheaves on $\mathcal{X}$.

Although it makes sense to talk of $\mathcal{C}$-valued sheaves for any ∞-category $\mathcal{C}$, most elementary constructions require additional hypotheses on $\mathcal{C}$ such as the existence of limits or presentability. The following proposition shows that in the presence of such hypotheses, the theory of $\mathcal{C}$-valued sheaves is reasonable.

Proposition 4.1.5. *Let $\mathcal{X}$ be an ∞-topos and $\mathcal{C}$ a presentable ∞-category, and let $\mathrm{Shv}(\mathcal{X}, \mathcal{C})$ denote the ∞-category of $\mathcal{C}$-valued sheaves on $\mathcal{X}$.*

1. *The inclusion functor $\mathrm{Shv}(\mathcal{X}, \mathcal{C}) \subseteq \mathcal{C}^{\mathcal{X}^{op}}$ admits a left adjoint.*

2. *The ∞-category $\mathrm{Shv}(\mathcal{X}, \mathcal{C})$ is presentable.*

3. *Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a geometric morphism of ∞-topoi. Then $f_* : \mathrm{Shv}(\mathcal{X}, \mathcal{C}) \to \mathrm{Shv}(\mathcal{Y}, \mathcal{C})$ has a left adjoint f^*.*

The proof is a technical bit of ∞-category theory in the spirit of the second section of [22]. We will sketch it for completeness, but it may be skipped without loss of continuity.

Proof. Choose a regular cardinal κ so that $\mathcal{X} = \mathrm{Ind}_\kappa(\mathcal{X}_\kappa)$, where $\mathcal{X}_\kappa$ denotes the full subcategory of $\mathcal{X}$ consisting of κ-compact objects. Since $\mathcal{X}_\kappa$ generates $\mathcal{X}$ under colimits, we have fully faithful inclusions

$$\mathrm{Shv}(\mathcal{X}, \mathcal{C}) \subseteq \mathcal{C}^{\mathcal{X}_\kappa^{op}} \subseteq \mathcal{C}^{\mathcal{X}^{op}},$$

where the second inclusion identifies $\mathcal{C}^{\mathcal{X}_\kappa^{op}}$ with the ∞-category of $\mathcal{C}$-valued presheaves on $\mathcal{X}$ which are compatible with κ-filtered colimits. The second inclusion has a left adjoint, given by restriction to $\mathcal{X}_\kappa$. Since $\mathcal{X}_\kappa$ is essentially small, the middle ∞-category is presentable. To complete the proofs of (1) and (2), it will suffice to show that the right-hand inclusion admits a left adjoint L, which is accessible when regarded as an endofunctor of $\mathcal{C}^{\mathcal{X}_\kappa^{op}}$. For κ sufficiently large, this inclusion is simply the pushforward along the geometric morphism $\mathcal{X} \to \mathcal{S}^{\mathcal{X}_\kappa^{op}}$, so it will suffice to prove (3) in the special case where $\mathcal{Y} = \mathcal{X}_\kappa^{op}$.

A functor $\mathcal{F} : \mathcal{X}_\kappa \to \mathcal{C}^{op}$ belongs to $\mathcal{D}$ if and only if it commutes with all κ-small colimits in $\mathcal{X}_\kappa$. Since $\mathcal{X}_\kappa$ is essentially small, the collection of such diagrams is bounded in size. The functor L may be constructed by a standard transfinite procedure which forces $\mathcal{F}$ to be compatible with every such diagram. We leave the details to the reader.

We now prove (3) in general. Suppose first that $\mathcal{C} = \mathcal{S}$ is the ∞-category of spaces. Then $\mathrm{Shv}(\mathcal{X}, \mathcal{C})$ is equivalent to $\mathcal{X}$. Assertion (3) follows from the definition of a geometric morphism.

If $\mathcal{C}$ is the ∞-category of presheaves on some small ∞-category $\mathcal{C}_0$, then the existence and accessibility of the left adjoint of (3) may be proven by working componentwise.

In the general case, we may realize the presentable ∞-category $\mathcal{C}$ as the essential image of some localization functor $L : \mathcal{P} \to \mathcal{P}$, where $\mathcal{P}$ is an ∞-category of presheaves. We have already established the existence of $f^* : \mathrm{Shv}(\mathcal{Y}, \mathcal{P}) \to \mathrm{Shv}(\mathcal{Y}, \mathcal{P})$. The pullback functor is defined on $\mathrm{Shv}(\mathcal{Y}, \mathcal{C})$ by applying f^* to obtain an object of $\mathrm{Shv}(\mathcal{X}, \mathcal{P})$, localizing it to obtain a $\mathcal{C}$-valued presheaf on $\mathcal{X}_\kappa$, and then sheafifying this presheaf. □

Remark 4.1.6. Proposition 4.1.5 is extremely formal, and never really used the fact that we are dealing with sheaves on ∞-topoi. The price, of course, is that the existence of a left adjoint tells us very little about how to compute it.

We can say much more about the theory of $\mathcal{C}$-valued sheaves if we impose further conditions on $\mathcal{C}$. Let us call an ∞-category $\mathcal{C}$ *compactly presented* if it is presentable and generated by its compact objects. Equivalently, $\mathcal{C}$ is presentable if and only if there is an equivalence $\mathcal{C} \simeq \mathrm{Ind}(\mathcal{C}_0)$, where $\mathcal{C}_0$ is an essentially small ∞-category which admits finite colimits. In fact, we may take $\mathcal{C}_0$ to be the subcategory of all compact objects of $\mathcal{C}$.

If $\mathcal{C}$ is a fixed ∞-category, then we shall refer to a pair $(\mathcal{X}, \mathcal{O})$ consisting of an ∞-topos $\mathcal{X}$ and a $\mathcal{C}$-valued sheaf $\mathcal{O}$ on $\mathcal{X}$ as a *$\mathcal{C}$-structured ∞-topos.* A morphism $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ of $\mathcal{C}$-structured ∞-topoi consists of a pair (f, ϕ) where $f : \mathcal{X} \to \mathcal{Y}$ is a geometric morphism and $\phi : \mathcal{O}_{\mathcal{Y}} \to f_* \mathcal{O}_{\mathcal{X}}$ is a morphism of $\mathcal{C}$-valued sheaves on $\mathcal{Y}$. A morphism (f, ϕ) is said to be *étale* if f is an étale morphism and the adjoint map $f^* \mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}$ is an equivalence (it is easy to see that the adjoint f^* always exists when f is étale: it is simply given by restriction). The $\mathcal{C}$-structured ∞-topoi form a $(\infty, 2)$-category. In general, there is no reason to expect the morphism ∞-categories $\mathrm{Hom}((\mathcal{X}, \mathcal{O}_{\mathcal{X}}), (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}))$ to be small (this is not even true when $\mathcal{C}$ is trivial). However, these morphism categories are always accessible when $\mathcal{C}$ is presentable (this may be proven using straightforward cardinality estimates).

Proposition 4.1.7. *Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor between compactly presented ∞-categories. Suppose that F preserves all colimits. Then F has a left adjoint G. We may regard G as also defined on $\mathcal{C}'$-valued (pre)sheaves on ∞-topoi. If $G : \mathcal{C}' \to \mathcal{C}$ commutes with filtered colimits, then the functor G commutes with sheafification and with pullback along geometric morphisms.*

Proof. The existence of the adjoint G follows from the adjoint functor theorem. Let $\mathcal{X}$ be an ∞-topos. The sheafification of a presheaf may be obtained by first restricting the presheaf to $\mathcal{X}_\kappa$ for a large regular cardinal κ, and then applying the pullback along the natural geometric morphism $\mathcal{X} \to \mathcal{S}^{\mathcal{X}_\kappa^{op}}$. Since G clearly commutes with restriction, it will suffice to prove that G commutes with pullbacks.

Let $\mathcal{C}_0$ denote the full subcategory $\mathcal{C}$ consisting of compact objects. Then $\mathcal{C}$ is equivalent to the ∞ category of all functors $\mathcal{C}_0^{op} \to \mathcal{S}$ which preserve finite limits. Consequently, we see that a $\mathcal{C}$-valued presheaf on an ∞-topos $\mathcal{Y}$ may be considered as a functor $\mathcal{Y}^{op} \times \mathcal{C}_0^{op} \to \mathcal{S}$, which preserves all limits in the first variable, and finite limits in the second variable. This is equivalent to the category of left-exact functors $\mathcal{O} : \mathcal{C}_0^{op} \to \mathcal{Y}$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi. We may then define $f^* \mathcal{O}$ to be the composite functor $f^* \circ \mathcal{O} : \mathcal{C}_0^{op} \to \mathcal{X}$, which remains left exact. It is easy to check that $f^* \mathcal{O}$ has the appropriate mapping property.

If G commutes with filtered colimits, then F carries compact objects of $\mathcal{C}$ into compact objects of $\mathcal{C}'$. If $\mathcal{O} : (\mathcal{C}'_0)^{op} \to \mathcal{Y}$ represents a $\mathcal{C}'$-valued presheaf on $\mathcal{Y}$, then $G\,\mathcal{O}$ is obtained by precomposition with $F|\,\mathcal{C}_0$, while $f^* \mathcal{O}$ is obtained by postcomposition with f^*. Since precomposition and postcomposition commute with one another, we deduce that f^* and G commute. □

Example 4.1.8. Let $F : \mathcal{S} \to \mathcal{SCR}$ be the "free algebra" functor, which is left adjoint to the "underlying space" functor G. Since G commutes with filtered colimits, we deduce that pullback and sheafification of $\mathcal{SCR}$-valued sheaves are compatible with passage to the underlying spaces.

In the proof of Proposition 4.1.7, we saw that to give a $\mathcal{C}$-valued sheaf on $\mathcal{X}$ is equivalent to giving a left-exact functor $\mathcal{C}_0^{op} \to \mathcal{X}$. Giving a functor $f : \mathcal{C}_0^{op} \to \mathcal{X}$ is equivalent to giving a colimit preserving functor $F : \mathcal{S}^{\mathcal{C}_0} \to \mathcal{X}$. Moreover, using the fact that $\mathcal{X}$ is an ∞-topos one sees that f is left exact if and only if F is left exact. In other words, the ∞-category of

$\mathcal{C}$-valued sheaves on $\mathcal{X}$ is equivalent to the ∞-category of geometric morphisms $F : \mathcal{X} \to \mathcal{S}^{\mathcal{C}_0}$. In particular, the identity functor from $\mathcal{S}^{\mathcal{C}_0}$ to itself gives rise to a universal $\mathcal{C}$-valued sheaf $\mathcal{O}$ on $\mathcal{S}^{\mathcal{C}_0}$. Any $\mathcal{C}$-valued sheaf $\mathcal{O}_{\mathcal{X}}$ on any ∞-topos $\mathcal{X}$ is equivalent to $F^* \mathcal{O}$ for some (essentially unique) geometric morphism $F : \mathcal{X} \to \mathcal{S}^{\mathcal{C}_0}$. Hence, we may say that $\mathcal{S}^{\mathcal{C}_0}$ is a *classifying ∞-topos* for $\mathcal{C}$-valued sheaves.

Of course, this is only the tip of the iceberg: just as for ordinary topoi, any ∞-topos may be interpreted as a classifying topos for a sufficiently complicated type of structure. However, we shall only need a slight generalization of the above discussion, modified to take into account a "topology" on the ∞-category $\mathcal{C}$.

Definition 4.1.9. Let $\mathcal{C}$ be an ∞-category with finite colimits. An *admissible topology* on $\mathcal{C}$ consists of the following data:

- A class of morphisms of $\mathcal{C}$ called *admissible morphisms.*
- For each object A in $\mathcal{C}$, a class of families $\{A \to A_\alpha\}$ of admissible morphisms in $\mathcal{C}$ called *covering families.*

These notions are required to satisfy the following conditions:

- Any morphism equivalent to an admissible morphism is admissible. Any family of morphisms equivalent to a covering family is also a covering family.
- Any identity $A \to A$ is admissible, and the one-element family $\{A \to A\}$ is covering.
- Any composition of admissible morphisms is admissible. If $\{A \to A_\alpha\}$ is a covering family and for each α, $\{A_\alpha \to A_{\alpha\beta}\}$ is also covering, then the composite family $\{A \to A_{\alpha\beta}\}$ is covering.
- If $A \to A'$ is an admissible morphism and $A \to B$ is arbitrary, then the pushout $B \to A' \coprod_A B$ is admissible. If a family $\{A \to A_\alpha\}$ of admissible morphisms is covering, and $A \to B$ is arbitrary, then the induced family $\{B \to B \coprod_A A_\alpha\}$ is covering.
- If a family $\{A \to A_\alpha\}$ of admissible morphisms is covering, then any larger family is also covering. Conversely, any covering family contains a finite subfamily which is also covering.

Remark 4.1.10. Several examples of an algebro-geometric nature will be given in the next section. The only example which will really concern us in this paper is the étale topology on $\mathcal{SCR}$. In this topology, the admissible morphisms are the étale morphisms, and the covering families are those which induce covering families in the classical sense after passing to the ordinary commutative rings of connected components.

Remark 4.1.11. If $\mathcal{C}$ is a small ∞-category, then an admissible topology on $\mathcal{C}$ determines a Grothendieck topology on the opposite ∞-category $\mathcal{C}^{op}$. More generally, for each $A \in \mathcal{C}$, we obtain a Grothendieck topology on the opposite of the ∞-category of admissible objects of $\mathcal{C}_{A/}$. We summarize the theory of Grothendieck topologies on ∞-categories in the appendix. For our applications it will be convenient to work with the slightly more structured notion of an admissible topology.

Example 4.1.12. For any ∞-category $\mathcal{C}$ with finite colimits, we may equip $\mathcal{C}$ with the *trivial topology*: the admissible morphisms are precisely the equivalences, and the covering families are those families which are nonempty.

Let $\mathcal{C}_0$ be a small category with finite colimits and an admissible topology $\mathcal{T}$, and let $\mathcal{C} = \mathrm{Ind}(\mathcal{C}_0)$. We will say that a morphism $A \to B$ in $\mathcal{C}$ is *admissible* if there exists an admissible morphism $A_0 \to B_0$ in $\mathcal{C}_0$ and a morphism $A_0 \to A$ which identifies B with the pushout $A \coprod_{A_0} B_0$. Similarly, we shall say that a family $\{A \to A_\alpha\}$ of admissible morphisms in $\mathcal{C}$ is a covering family if there exists a covering family $\{B \to B_\alpha\}$ in $\mathcal{C}_0$, a morphism $B \to A$, and identifications of each A_α with the pushout $A \coprod_B B_\alpha$. One can easily verify that this defines an admissible topology on $\mathcal{C}$ (note that the proof that admissible coverings compose requires the assumption that every covering has a finite refinement). An admissible topology on $\mathcal{C}$ is said to be *compactly generated* if it arises in this way.

Suppose that $\mathcal{C}$ is an ∞-category with finite colimits, $\mathcal{X}$ an ∞-topos, and $\mathcal{O}$ a $\mathcal{C}$-valued sheaf on $\mathcal{X}$. Let $U \in \mathcal{X}$ be an object and $\psi : \mathcal{O}(U) \to A$ an arbitrary morphism of $\mathcal{C}$. The functor

$$V \mapsto \mathrm{Hom}_{\mathcal{O}(U)}(A, \mathcal{O}(V))$$

from $\mathcal{X}_{/U}$ to $\mathcal{S}$ carries colimits into limits, and is therefore representable by an object of $\mathcal{X}_{/U}$ which we shall denote by $\mathrm{Sol}(\phi)$. The intuition is that the object A admits some "presentation" over $\mathcal{O}(U)$ by generators and relations, which we may think of as variables and equations. Then $\mathrm{Sol}(\phi)$ is the "space of solutions" to those equations in the structure sheaf $\mathcal{O}$.

Definition 4.1.13. Let $\mathcal{C}$ be an ∞-category with finite colimits and an admissible topology $\mathcal{T}$, and let $\mathcal{X}$ be an ∞-topos.

- A $\mathcal{C}$-valued sheaf $\mathcal{O}$ on $\mathcal{X}$ is $\mathcal{T}$-*local* if for any admissible covering $\{\psi_\alpha : \mathcal{O}(U) \to A_\alpha\}$, the family $\mathrm{Sol}(\psi_\alpha)$ forms a covering of U.
- A morphism $\mathcal{O} \to \mathcal{O}'$ of $\mathcal{C}$-valued sheaves is said to be $\mathcal{T}$-*local* if, for any $U \in \mathcal{X}$ and any admissible morphism $\psi : \mathcal{O}(U) \to A$, the natural map $\mathrm{Sol}(\psi) \to \mathrm{Sol}(\psi')$ is an equivalence, where $\psi' : \mathcal{O}'(U) \to A \coprod_{\mathcal{O}(U)} \mathcal{O}'(U)$ is the induced morphism.

More generally, given a morphism (f, ϕ) between $\mathcal{C}$-structured ∞-topoi $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$, we shall say that (f, ϕ) is $\mathcal{T}$-*local* if for any $U \in \mathcal{Y}$ and any admissible morphism $\phi : \mathcal{O}_{\mathcal{Y}}(U) \to A$, the induced map $f^* \mathrm{Sol}(\psi) \to \mathrm{Sol}(\psi')$ is an equivalence, where $\psi' : \mathcal{O}_{\mathcal{X}}(f^*U) \to \mathcal{O}_{\mathcal{X}}(f^*U) \coprod_{\mathcal{O}_{\mathcal{Y}}(U)} A$ is the cobase extension of ψ.

Before we can establish the basic properties of $\mathcal{T}$-locality, we need a couple of elementary lemmas.

Lemma 4.1.14. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi, $\mathcal{C}$ a compactly presented ∞-category, and $\mathcal{T}$ a compactly generated, admissible topology on $\mathcal{C}$. Let $\mathcal{O}$ be a $\mathcal{C}$-valued sheaf on $\mathcal{Y}$, $U \in \mathcal{X}$ an object, and $\{\psi_\alpha : (f^* \mathcal{O})(U) \to A_\alpha\}$ a finite collection of admissible morphisms of $\mathcal{C}$.*

1. *There exists a surjection $U' \to U$ in $\mathcal{X}$, an object $V \in \mathcal{Y}$, a map $p : U' \to f^*V$, a collection of admissible morphisms $\{\mathcal{O}(V) \to B_\alpha\}$, and equivalences (under $(f^* \mathcal{O})(U')$)*

$$B_\alpha \coprod_{\mathcal{O}(V)} (f^* \mathcal{O})(U') \simeq A_\alpha \coprod_{(f^* \mathcal{O})(U)} (f^* \mathcal{O})(U').$$

2. *If, furthermore, the family $\{(f^* \mathcal{O})(U) \to A_\alpha\}$ is covering for $\mathcal{T}$, then we may arrange that the family $\{\mathcal{O}(V) \to B_\alpha\}$ is covering for $\mathcal{T}$.*

Proof. We will be content to give the proof of the first part (the proof of the second part is analogous). Without loss of generality, it suffices to treat the case of a single admissible morphism $(f^* \mathcal{O})(U) \to A$. Since $\mathcal{T}$ is compactly generated, there exists an admissible morphism $B_0 \to A_0$ of compact objects of $\mathcal{C}$, a morphism $p : B_0 \to (f^* \mathcal{O})(U)$, and an identification of A with $A_0 \coprod_{B_0} (f^* \mathcal{O})(U)$. It will suffice to prove that after replacing U by some cover U', the morphism p is the pullback of some morphism defined over $\mathcal{Y}$. Since B_0 is compact, the sheaf

$$U \mapsto \mathrm{Hom}_{\mathcal{C}}(B_0, f^* \mathcal{O}(U))$$

is the pullback of the sheaf

$$V \mapsto \mathrm{Hom}_{\mathcal{C}}(B_0, \mathcal{O}(V)).$$

Consequently, we may reduce to the case where $\mathcal{C} = \mathcal{S}$, which is clear. □

Proposition 4.1.15. *Let $\mathcal{C}$ be a compactly presented ∞-category equipped with a compactly generated, admissible topology $\mathcal{T}$. Then:*

1. *The class of $\mathcal{T}$-local morphisms between $\mathcal{C}$-structured ∞-topoi is stable under equivalence.*

2. *Any equivalence of $\mathcal{C}$-structured ∞-topoi is $\mathcal{T}$-local.*

3. *Any composition of $\mathcal{T}$-local morphisms of $\mathcal{C}$-structured ∞-topoi is $\mathcal{T}$-local.*

4. *Let $\mathcal{X}$ be an ∞-topos, and $\phi : \mathcal{O} \to \mathcal{O}'$ be a $\mathcal{T}$-local morphism between $\mathcal{C}$-valued sheaves on $\mathcal{X}$. If $\mathcal{O}'$ is $\mathcal{T}$-local, then $\mathcal{O}$ is $\mathcal{T}$-local.*

5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi. Let $\mathcal{O}$ be a $\mathcal{C}$-valued sheaf on $\mathcal{Y}$. Then the induced morphism $(\mathcal{X}, f^* \mathcal{O}) \to (\mathcal{Y}, \mathcal{O})$ is $\mathcal{T}$-local.*

6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi. Let $\mathcal{O}$ be a $\mathcal{C}$-valued sheaf on $\mathcal{Y}$. If $\mathcal{O}$ is $\mathcal{T}$-local, then so is $f^* \mathcal{O}$.*

7. *Let $(f, \phi) : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a morphism of $\mathcal{C}$-structured spaces. Then (f, ϕ) is $\mathcal{T}$-local if and only if $\phi : f^* \mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}$ is $\mathcal{T}$-local.*

Proof. Assertions (1) through (4) are obvious (and do not require any compact generation assumptions). Assertion (5) follows from the fact that $\mathcal{T}$ is compactly generated and the fact that for $B \in \mathcal{C}$ compact, the correspondence

$$\mathcal{F} \mapsto (U \mapsto \mathrm{Hom}_{\mathcal{C}}(B, \mathcal{F}(U))$$

commutes with pullback. Assertion (6) is a little bit more subtle: suppose that $U \in \mathcal{X}$ and that we are given a covering family $\{\psi'_\alpha : (f^* \mathcal{O})(U) \to A'_\alpha\}$. We wish to prove that the family $\{\mathrm{Sol}(\psi'_\alpha)\}$ covers U. This assertion is local on U and on $\mathcal{Y}$, so we can use Lemma 4.1.14 to reduce to the case where $\{\psi'_\alpha\}$ is the cobase extension of a covering family $\{\psi_\alpha : \mathcal{O}(V) \to A_\alpha\}$. Since $\mathrm{Sol}(\psi'_\alpha) = f^* \mathrm{Sol}(\psi_\alpha)$ by (5), we get the desired result (using the fact that $(\mathcal{Y}, \mathcal{O})$ is $\mathcal{T}$-local).

It remains to prove (7). The "if" direction follows from (5) and (3). For the reverse direction we again reduce to the local case and apply Lemma 4.1.14. □

Definition 4.1.16. Let $\mathcal{C}$ be a compactly presented ∞-category with a compactly generated admissible topology $\mathcal{T}$. Let $(\mathcal{X}, \mathcal{O})$ be a $\mathcal{C}$-structured space. A *spectrum* for $(\mathcal{X}, \mathcal{O})$ is a $\mathcal{T}$-local, $\mathcal{C}$-structured ∞-topos $\mathrm{Spec}(\mathcal{X}, \mathcal{O})$ together with a morphism $f : \mathrm{Spec}(\mathcal{X}, \mathcal{O}) \to (\mathcal{X}, \mathcal{O})$ which possesses the following universal property: for any $\mathcal{T}$-local $(\mathcal{X}', \mathcal{O}')$, composition with f induces an equivalence of ∞-categories

$$\mathrm{Hom}_{\mathcal{T}}((\mathcal{X}', \mathcal{O}'), \mathrm{Spec}(\mathcal{X}, \mathcal{O})) \to \mathrm{Hom}((\mathcal{X}', \mathcal{O}'), (\mathcal{X}, \mathcal{O})).$$

Here the subscript on the left hand side indicates that we consider only $\mathcal{T}$-local morphisms.

The remainder of this section is devoted to proving the existence of spectra for $\mathcal{C}$-structured ∞-topoi $(\mathcal{X}, \mathcal{O})$. In order to simplify the discussion, we will restrict our attention to the case in which $\mathcal{X}$ is an ∞-topos of presheaves on some small ∞-category. This covers the only case that we will really use later: namely, $\mathcal{X} = \mathcal{S}$. We note that if $f : \mathcal{X} \to \mathcal{Y}$ is a geometric morphism, $\mathcal{O}$ is a $\mathcal{C}$-valued sheaf on $\mathcal{Y}$, and $(\mathcal{Y}', \mathcal{O}')$ is a spectrum for $(\mathcal{Y}, \mathcal{O})$, then the lax fiber product $(\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}, (f')^* \mathcal{O}')$ is a spectrum for $(\mathcal{X}, f^* \mathcal{O})$, provided that the fiber product exists. Here f' denotes the projection onto the first factor. The construction of these lax fiber products is treated in [22]. We only wish to note that, granting their existence, the problem of constructing spectra can be reduced to the universal case where $\mathcal{X}$ is the classifying topos for $\mathcal{C}$-valued sheaves, which is again an ∞-topos of presheaves. In other words, the special case that we are treating here is really quite general.

So let us now suppose that $\mathcal{X} = \mathcal{S}^{\mathcal{D}^{op}}$ is the ∞-topos of presheaves on $\mathcal{D}$, where $\mathcal{D}$ is a small ∞-category. A $\mathcal{C}$-valued sheaf on $\mathcal{X}$ may be identified with a $\mathcal{C}$-valued presheaf $\mathcal{O} : \mathcal{D}^{op} \to \mathcal{C}$. From this data we shall give an explicit construction of $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) = \mathrm{Spec}(\mathcal{X}, \mathcal{O})$.

Let $\widetilde{\mathcal{D}}$ be the ∞-category of pairs (D, A) where $D \in \mathcal{D}$ and $\psi : \mathcal{O}(D) \to A$ is an admissible morphism (we will omit ψ from the notation but it is part of the data). A morphism $(D, A) \to (D', A')$ consists of a morphism $D \to D'$ and an extension of the composite map $\mathcal{O}(D') \to \mathcal{O}(D) \to A$ to a map $A' \to A$.

We equip $\widetilde{\mathcal{D}}$ with a Grothendieck topology by declaring a family of morphisms to be covering if and only if it contains a family equivalent to $\{(D, A_\alpha) \to (D, A)\}$, where the family of morphisms $(A \to A_\alpha)$ is an admissible covering of A for the topology $\mathcal{T}$. We then define $\mathcal{Y}$ to be the ∞-category of $\mathcal{S}$-valued sheaves on $\widetilde{\mathcal{D}}$.

Remark 4.1.17. For a discussion of $\mathcal{S}$-valued sheaves on ∞-categorical sites, we refer the reader to the appendix. We warn the reader that although our notion of an ∞-category with a Grothendieck topology is equivalent to that of [39], our notion of a sheaf is different since we impose weaker descent conditions.

Let $\mathcal{O}_\mathcal{Y}$ denote the sheafification of the $\mathcal{C}$-valued presheaf $\widetilde{\mathcal{O}}_\mathcal{Y}$ given by

$$\widetilde{\mathcal{O}}_\mathcal{Y}(D, A) = A.$$

Proposition 4.1.18. *The $\mathcal{C}$-structured space $(\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ is a spectrum for $(\mathcal{X}, \mathcal{O})$.*

Proof. We note that there is a functor $\mathcal{D} \to \widetilde{\mathcal{D}}$, given by

$$D \mapsto (D, \mathcal{O}(D)).$$

This functor gives a continuous map of sites, where $\mathcal{D}$ is equipped with the discrete topology. Consequently, it induces a geometric morphism $f : \mathcal{Y} \to \mathcal{X}$. Then $f_* \mathcal{O}_\mathcal{Y}$ is given by

$$D \mapsto \mathcal{O}_\mathcal{Y}(D, \mathcal{O}(D)).$$

In particular, there is a morphism $\mathcal{O}(D) \to (f_* \mathcal{O}_\mathcal{Y})(D)$ which is natural in D. The adjoint morphism $\phi : f^* \mathcal{O} \to \mathcal{O}_\mathcal{Y}$, together with f, give a morphism $(\mathcal{Y}, \mathcal{O}_\mathcal{Y}) \to (\mathcal{X}, \mathcal{O})$ of $\mathcal{C}$-structured ∞-topoi.

We next show that $(\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ is $\mathcal{T}$-local. Suppose that $U \in \mathcal{Y}$, and that $\{\psi'_\alpha : \mathcal{O}_\mathcal{Y}(U) \to A'_\alpha\}$ is a covering family. We must show that the family $\{\mathrm{Sol}(\psi'_\alpha)\}$ covers U. Without loss of generality, we may suppose that the family is finite. We also note that the assertion is local on U. Consequently, we may suppose that U is the sheafification of the presheaf on $\widetilde{\mathcal{D}}$ represented by an object (D, A), that each ψ'_α is the base change of some admissible morphism $\psi_\alpha : \widetilde{\mathcal{O}_\mathcal{Y}}(U) \to A_\alpha$, and that $\{\psi_\alpha\}$ is a covering family for $\widetilde{\mathcal{O}}_\mathcal{Y}(U) = A$. In this case, $\mathrm{Sol}(\psi'_\alpha)$ is the sheafification of the presheaf represented by (D, A_α), and these form a covering of (D, A) by construction.

It is now clear that the pair (f, ϕ) induces by composition a functor

$$F : \mathrm{Hom}_\mathcal{T}((\mathcal{Z}, \mathcal{O}_\mathcal{Z}), (\mathcal{Y}, \mathcal{O}_\mathcal{Y})) \to \mathrm{Hom}((\mathcal{Z}, \mathcal{O}_\mathcal{Z}), (\mathcal{X}, \mathcal{O})).$$

To complete the proof that $(\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ is a spectrum for $(\mathcal{X}, \mathcal{O})$, it suffices to show that F is an equivalence whenever $(\mathcal{Z}, \mathcal{O}_\mathcal{Z})$ is $\mathcal{T}$-local. We give a sketch of the construction of the inverse functor.

Suppose we are given a geometric morphism $g : \mathcal{Z} \to \mathcal{X}$ and a transformation $\mathcal{O} \to g_* \mathcal{O}_\mathcal{Z}$. We may identify g with a left-exact functor $g_0 : \mathcal{D} \to \mathcal{Z}$. We then define $\widetilde{g_0}$ on $\widetilde{\mathcal{D}}$ by the formula $\widetilde{g_0}(D, A) = \mathrm{Sol}(\psi')$, where $\psi' : \mathcal{O}_\mathcal{Z}(g_0(D)) \to A'$ is the cobase extension of $\psi : \mathcal{O}(D) \to A$. One shows that $\widetilde{g_0}$ is left exact, so that it induces a geometric morphism $\widetilde{g}$ from $\mathcal{Z}$ to the ∞-topos of presheaves on $\widetilde{\mathcal{D}}$. By construction it extends naturally to a morphism of $\mathcal{C}$-structured spaces

$$(\mathcal{Z}, \mathcal{O}_\mathcal{Z}) \to (\mathcal{S}^{\widetilde{\mathcal{D}}^{op}}, \widetilde{\mathcal{O}}_\mathcal{Y}).$$

To complete the proof, it suffices to show that $\widetilde{g}$ factors through $(\mathcal{Y}, \mathcal{O}_\mathcal{Y})$. This is a purely "topological" assertion, having nothing to do with the structure sheaves. It follows from the universal property of $\mathcal{Y}$ (see Appendix 9), provided that we can show that $\widetilde{g_0}$ carries covering families in $\widetilde{D}$ into covering families in $\mathcal{Z}$. But this is precisely the condition that $\mathcal{Z}$ be $\mathcal{T}$-local. □

Remark 4.1.19. Let us say that an ∞-category is an *n-category* if $\mathrm{Hom}(X, Y)$ is $(n-1)$-truncated for any pair of objects X and Y. According to this definition, a 1-category is simply a category in the usual sense, and a 0-category is a partially ordered set. From the construction, we see that if $\mathcal{D}$ is an n-category and if the ∞-category of admissible objects under $\mathcal{O}(D)$ is an n-category for any $D \in \mathcal{D}$, then $\mathcal{Y}$ is an ∞-category of $\mathcal{S}$-valued sheaves on the n-category $\widetilde{D}$. If $n = 1$, this means that $\mathcal{Y}$ is the ∞-category associated to a topos; if $n = 0$, this means that $\mathcal{Y}$ is the ∞-category associated to a locale.

4.2 Topologies on Simplicial Commutative Rings

In this section, we will give various examples of admissible topologies on the ∞-category $\mathcal{SCR}$, which are derived analogues of topologies of interest in classical algebraic geometry. The only topology that we will really use later is the étale topology, but we feel that giving the general picture is helpful for clarifying the dependence of our formalism on a particular topology.

We will be concerned with the following examples:

1. $\mathcal{T}_{\text{ét}}$: The admissible morphisms are the étale morphisms. The covering families $\{R \to R_\alpha\}_{\alpha \in A}$ are those for which there exists a finite subset $A_0 \subseteq A$ such that

$$R' = \prod_{\alpha \in A_0} R_\alpha$$

is faithfully flat over R. We will refer to this as the *étale topology.*

2. $\mathcal{T}_{\text{Nis}}$: The admissible morphisms are the étale morphisms. The covering families are those for which $\{R \to R_\alpha\}_{\alpha \in A}$ be those for which there exists a finite sequence $R_1, \ldots, R_n$ taken from the R-algebras $\{R_\alpha\}$ and a finite sequence of compact open subsets $\emptyset = U_0 \subseteq U_1 \subseteq \ldots \subseteq U_n = X$ of the Zariski spectrum X of $\pi_0 R$, such that the (ordinary) scheme $\text{Spec}(\pi_0 R_i) \times_X (U_i - U_{i-1})$ contains an open subscheme which maps isomorphically to $U_i - U_{i-1}$. We shall refer to this as the *Nisnevich topology.*

3. $\mathcal{T}_{\text{Zar}}$: The admissible morphisms are the étale morphisms $f : R \to R'$ for which $\pi_0 f : \pi_0 R \to \pi_0 R'$ induces an open immersion of affine schemes. The covering families $\{R \to R_\alpha\}_{\alpha \in A}$ are those for which there exists a finite subset $A_0 \subseteq A$ such that

$$R' = \prod_{\alpha \in A_0} R_\alpha$$

is faithfully flat over R. We will refer to this as the *Zariski topology.*

4. $\mathcal{T}_{\text{triv}}$: The admissible morphisms are the equivalences. An admissible family $\{R \to R_\alpha : \alpha \in A\}$ is covering exactly when A is nonempty. We shall refer to this as the *trivial topology.*

It is easy to see that each of these examples satisfies the conditions of Definition 4.1.9. Moreover, the structure theory of étale morphisms shows that each of these topologies is compactly generated.

Remark 4.2.1. Each of the topologies defined above induces topologies on the ∞-category of admissible A-algebras, for any fixed $A \in \mathcal{SCR}$. Theorem 3.4.13 shows that this ∞-category is actually an ordinary category, since it is equivalent to a full subcategory of the étale $\pi_0 A$-modules. Moreover, the induced topology on the category of admissible A-algebras agrees with the same topology on the (equivalent) category of admissible $\pi_0 A$-algebras, which is simply the classical étale, Zariski, Nisnevich (with slight modifications), or trivial topology.

Remark 4.2.2. Our definition of the Nisnevich topology is slightly nonstandard. One usually declares that a family of étale morphisms $\{A \to A_\alpha\}$ is a Nisnevich covering if, for any residue field κ of A, some fiber $A_\alpha \otimes_A \kappa$ contains a factor isomorphic to κ. This definition is equivalent to ours if A is Noetherian (see Proposition 4.4.1), but the definition given above seems to have better formal properties in the general case. For example, the Nisnevich topology, as we have defined it, is compactly generated. In fact, our definition is uniquely prescribed by the requirements that the Nisnevich topology be compactly generated, and that it should agree with the usual Nisnevich topology in the Noetherian case.

If $\mathcal{T} = \mathcal{T}_{\text{ét}}$ or $\mathcal{T}_{\text{Nis}}$, and $R \in \mathcal{SCR}$, then the ∞-category of admissible R-algebras is stable under finite products. A family $\{R \to R_\alpha\}_{\alpha \in A}$ is covering if and only if there exists a finite subset $A_0 \subseteq A$ such that the single morphism $R \to \prod_{\alpha \in A_0} R_\alpha$ is covering. In these cases, the sheaf condition is easily stated in the language of rings. Namely, a presheaf on the category of admissible R-algebras is a sheaf if and only if it satisfies the following conditions:

- For any finite family of objects $\{R_\alpha\}$ of $\mathcal{C}$, the natural map $\mathcal{F}(\prod R_\alpha) \to \prod \mathcal{F}(R_\alpha)$ is an equivalence.

- For any covering morphism $S \to S_0$, if we form the simplicial object $S_\bullet$ of $\mathcal{C}$ with $S_n = S_0 \otimes_S \ldots \otimes_S S_0$ (n factors), then the natural map $\mathcal{F}(S) \to |\mathcal{F}(S_\bullet)|$ is an equivalence.

Remark 4.2.3. For the Zariski and étale topology, the notion of a covering satisfies flat descent in the following sense: if S is a faithfully flat R algebra, then a family of morphisms $\{R \to R_\alpha\}$ is covering if and only if the family $\{S \to R_\alpha \times_R S\}$ is covering. For the Nisnevich topology this fails, even if S is étale over R.

In addition to the four topologies defined above, it is occasionally useful to consider a much finer topology, the *flat hypertopology*. To introduce this, we need some notation for describing cosimplicial objects in ∞-categories. Let Δ denote the ordinary category of *combinatorial simplices*. The objects of Δ are finite, nonempty linearly ordered sets, and the morphisms are nondecreasing functions. We let $\Delta_{\leq n}$ denote the full subcategory of Δ consisting of simplices having dimension $\leq n$ (in other words, linearly ordered sets having cardinality $\leq n+1$). If $\mathcal{C}$ is any ∞-category, then a *cosimplicial object* of $\mathcal{C}$ is defined to be a functor $\Delta \to \mathcal{C}$. We will write cosimplicial objects of $\mathcal{C}$ as $C^\bullet$, where C^n denotes the evaluation of the cosimplicial object on the object $[0, \ldots, n] \in \Delta$.

An n-skeleton in $\mathcal{C}$ is defined to be a functor $\Delta_{\leq n} \to \mathcal{C}$. Restriction induces a functor from cosimplicial objects in $\mathcal{C}$ to n-skeletons in $\mathcal{C}$. If $\mathcal{C}$ has all finite colimits, then this functor has a right adjoint (which is constructed by a standard procedure). If $X^\bullet$ is a cosimplicial object of $\mathcal{C}$, we let $\operatorname{cosk}^n X^\bullet$ denote the result of applying this adjoint to the restriction of $X^\bullet$. Thus, there is an adjunction morphism

$$X^\bullet \leftarrow \operatorname{cosk}^n X^\bullet$$

which is an equivalence when evaluated on simplices of dimension $\leq n$.

We specialize to the case where $\mathcal{C} = \mathcal{SCR}_{/A}$ is the ∞-category of A-algebras, for some fixed $A \in \mathcal{SCR}$. This ∞-category has all finite colimits, so that we can construct coskeleta. A cosimplicial object $B^\bullet$ of $\mathcal{C}$ is called a *flat hypercovering* of A if the following condition is satisfied:

- For each $n \geq 0$, the adjunction $(\operatorname{cosk}^{n-1} B^\bullet)^n \to B^n$ is faithfully flat.

In other words, B^0 is faithfully flat over A, B^1 is faithfully flat over $B^0 \otimes_A B^0$, and so forth.

For any ∞-category $\mathcal{C}$, a functor $\mathcal{F} : \mathcal{SCR} \to \mathcal{C}$ is said to be a *sheaf for the flat hypertopology* if it satisfies the following conditions:

- For any finite collection of objects $\{A_i\}$ in $\mathcal{SCR}$, the natural map $\mathcal{F}(\Pi_i A_i) \to \Pi_i \mathcal{F}(A_i)$ is an equivalence.

- The natural map $\mathcal{F}(A) \to |\mathcal{F}(B^\bullet)|$ is an equivalence whenever $B^\bullet$ is a flat hypercovering of A.

Example 4.2.4. Let $\mathcal{F}$ denote the identity functor from $\mathcal{SCR}$ to itself. Then $\mathcal{F}$ is a $\mathcal{SCR}$-valued sheaf on $\mathcal{SCR}$. In other words, the flat hypertopology is a *sub-canonical* topology on $\mathcal{SCR}$. The main point is to show that if $A^\bullet$ is a flat hypercovering of $A \in \mathcal{SCR}$, then the natural map $A \to |A^\bullet|$ is an equivalence. For this, we use the Bousfield-Kan spectral sequence (see, for example, [6]) to compute the homotopy groups of $|A^\bullet|$. This spectral sequence has $E_{pq}^2 = \pi^{-p}(\pi_q A^\bullet) \Rightarrow \pi_{p+q}|A^\bullet|$, where $\pi^{-p}(\pi_q A^\bullet)$ indicates the $(-p)$th cohomotopy group of the cosimplicial abelian group $\pi_q A^\bullet$. Since $A^\bullet$ is a flat hypercovering of A, we deduce that $\pi^{-p}(\pi_q A^\bullet) = \pi_q A$ for $p = 0$ and vanishes otherwise. Thus the Bousfield-Kan spectral sequence degenerates at E_2 and demonstrates that $A \simeq |A^\bullet|$.

Example 4.2.5. Let $\mathcal{C}$ be the (very large) ∞-category of ∞-categories (where morphisms are given by functors, and we disregard non-invertible natural transformations of functors). Let $\mathcal{F} : \mathcal{SCR} \to \mathcal{C}$ assign to each $A \in \mathcal{SCR}$ the ∞-category $\mathcal{M}_A$ of A-modules. Then $\mathcal{F}$ is a sheaf for the flat hypertopology. It is easy to see that $\mathcal{F}$ carries finite products into finite products; the main point is to show that if $A \to A^\bullet$ is a flat hypercovering, then $\mathcal{M}_A$ is the geometric realization of the cosimplicial ∞-category $\mathcal{M}_{A^\bullet}$. This geometric realization may be interpreted as an ∞-category of *cosimplicial modules* over the cosimplicial object $A^\bullet$ of $\mathcal{SCR}$. Let $F : \mathcal{M}_A \to \mathcal{M}_{A^\bullet}$ denote the natural functor. Then F has a right adjoint G: one can either observe that F preserves limits and that both $\mathcal{M}_A$ and $\mathcal{M}_{A^\bullet}$ are presentable, or argue directly by setting $G(M^\bullet) = |M^\bullet|$. Now it suffices to show that the adjunction maps $M \to GFM$ and $FGN^\bullet \to N^\bullet$ are equivalences, for any $M \in \mathcal{M}_A$ and $N^\bullet \in \mathcal{M}_{A^\bullet}$. Both of these results follow from easy computations with the appropriate Bousfield-Kan spectral sequences (which degenerate at E_2).

Remark 4.2.6. The flat hypertopology also makes sense in the context of connective A_∞-ring spectra. Examples 4.2.4 and 4.2.5 generalize easily to this setting.

Remark 4.2.7. Descent for modules as formulated in Example 4.2.5 is only a prototype for a host of similar results. All manners of variations (such as descent for algebras) may be established in the same manner.

Remark 4.2.8. We could also define étale, Nisnevich, and Zariski hypertopologies, as well as a "flat topology" which imposed descent only for 1-coskeletal flat hypercoverings. However, we shall not need these intermediate notions.

Remark 4.2.9. The flat hypertopology is very much unlike the other topologies considered in this section, for the following reasons:

- There is no strong relationship between the flat hypertopology on the ∞-category of flat A-algebras and the flat hypertopology on the category of flat $\pi_0 A$-algebras.

- In the context of the flat hypertopology, one considers *all* A-algebras, rather than simply some class of admissible A-algebras. The ∞-category $\mathcal{SCR}_{A/}$ is not small, so one does not expect an ∞-topos of sheaves for the flat hypertopology or any kind of "sheafification" functor.

- The functor represented by a derived scheme (see §4.5 for the definition) need not be a sheaf for the flat hypertopology. (For example, a general result of this type would imply the equivalence of flat and étale cohomology with arbitrary coefficients.) However, most of the derived schemes and derived stacks which arise naturally do satisfy this stronger descent condition. A very general descent result of this type will be proven in [24].

4.3 Spectra of Simplicial Commutative Rings

An ordinary scheme is defined to be a topological space with a sheaf of rings, which is locally isomorphic to some affine model $\operatorname{Spec} A$ with its Zariski topology, where A is some commutative ring. Our definition will have the same form, but will differ in certain respects:

- The Zariski topology, while sufficient for many applications, provides an inadequate foundation for describing algebraic spaces and Deligne-Mumford stacks. For this reason, we will employ the étale topology in place of the Zariski topology.

- The Zariski topos of a commutative ring is localic and has enough points, and may therefore be adequately described in terms of a topological space. However, the étale and Nisnevich topoi of a commutative ring are not localic, and in order to use these topologies in a serious way one must replace the notion of a "ringed space" with that of a "ringed topos". Although it is not strictly necessary to go any further than this (see Theorem 4.5.10), it will be convenient to formulate our definition in terms of "ringed ∞-topoi".

- We will allow our local models to have the form $\operatorname{Spec} A$, where $A \in \mathcal{SCR}$, rather than restricting our attention to discrete commutative rings. We remark that this generalization is completely independent of the topological considerations described above, since the étale topology of $A \in \mathcal{SCR}$ is identical with that of $\pi_0 A$.

We shall abuse terminology by saying that a $\mathcal{SCR}$-valued sheaf $\mathcal{O}$ on $\mathcal{X}$ is a *sheaf of rings* on $\mathcal{X}$. A *ringed* ∞-topos is a pair consisting of an ∞-topos $\mathcal{X}$ and a $\mathcal{SCR}$-valued sheaf $\mathcal{O}$ on $\mathcal{X}$. In the terminology of §4.1, we may say that a ringed ∞-topos is a is a $\mathcal{SCR}$-structured ∞-topos. Morphisms between ringed ∞-topoi are defined to be morphisms of $\mathcal{SCR}$-structured ∞-topoi. We remark that the collection of morphisms between two ringed ∞-topoi is naturally organized into an ∞-category, so that the ringed ∞-topoi themselves constitute an $(\infty, 2)$-category. If (f, ϕ) is a morphism of ringed ∞-topoi, we will typically abuse notation and simply refer to f, with ϕ being understood.

We now discuss the relationship between $\mathcal{SCR}$-valued sheaves and sheaves of ordinary commutative rings. More generally, we will consider n-truncated rings for any $n \geq 0$.

Definition 4.3.1. A $\mathcal{SCR}$-valued sheaf $\mathcal{O}$ on an ∞-topos $\mathcal{X}$ is *n-truncated* if $\mathcal{O}(U)$ is n-truncated for any $U \in \mathcal{X}$. In this case we shall also say that $(\mathcal{X}, \mathcal{O})$ is n-truncated.

Remark 4.3.2. In order to check that a $\mathcal{SCR}$-valued sheaf $\mathcal{O}$ is n-truncated, it suffices to verify that $\mathcal{O}(U)$ is n-truncated as U ranges over a family of objects which generates the ∞-topos $\mathcal{X}$ under colimits. For example, if $\mathcal{X}$ is the ∞-topos of sheaves on an ordinary topos $\mathfrak{X}$, then it suffices to check that $\mathcal{O}(U)$ is n-truncated for $U \in \mathfrak{X}$.

Remark 4.3.3. Let $F = \tau_{\leq n} : \mathcal{SCR} \to \tau_{\leq n}\mathcal{SCR}$ be the localization, which is left adjoint to the inclusion $G : \tau_{\leq n}\mathcal{SCR} \subseteq \mathcal{SCR}$. Since G commutes with filtered colimits, Proposition 4.1.7 implies that it is safe to ignore the distinction between $\mathcal{SCR}$-valued sheaves which take n-truncated values, and $\tau_{\leq n}\mathcal{SCR}$-valued sheaves. In particular, when $n = 0$, we see that the notion of a $\mathcal{SCR}$-valued sheaf on an ∞-topos $\mathcal{X}$ is an honest generalization of the notion of a sheaf of commutative rings on $\mathcal{X}$ (which is the same as a commutative ring object in the ordinary topos $\tau_{\leq 0}\mathcal{X}$).

If $\mathcal{O}$ is a sheaf of rings on an ∞-topos $\mathcal{X}$, then we let $\tau_{\leq n}\mathcal{O}$ denote the $\mathcal{SCR}$-valued sheaf on $\mathcal{X}$ which is obtained by sheafifying the presheaf

$$U \mapsto \tau_{\leq n}\mathcal{O}(U).$$

Proposition 4.1.7 implies that $(\mathcal{X}, \tau_{\leq n}\mathcal{O}_{\mathcal{X}})$ is n-truncated. There is a natural map $f : (\mathcal{X}, \tau_{\leq n}\mathcal{O}) \to (\mathcal{X}, \mathcal{O})$. By construction, f is universal with respect to these properties, in the sense that any morphism from an n-truncated ringed ∞-topos $(\mathcal{X}', \mathcal{O}')$ to $(\mathcal{X}, \mathcal{O})$ factors uniquely through $(\mathcal{X}, \tau_{\leq n}\mathcal{O})$.

Proposition 4.3.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi, and let $\mathcal{O}$ be a $\mathcal{SCR}$-valued sheaf on $\mathcal{Y}$. Then there is a natural equivalence $\tau_{\leq n} f^*\mathcal{O} \simeq f^*\tau_{\leq n}\mathcal{O}$. In particular, if $\mathcal{O}$ is n-truncated, then so is $f^*\mathcal{O}$.*

Proof. One first shows that the functor $\tau_{\leq n}$ is compatible with the "underlying space" functor from $\mathcal{SCR}$-valued sheaves to $\mathcal{S}$-valued sheaves. It then follows that f^* preserves the property of being n-truncated. The universal property of $\tau_{\leq n}$ then produces a natural transformation

$$f^*\tau_{\leq n} \to \tau_{\leq n} f^*.$$

To see that this natural transformation is an equivalence, it suffices to check on the underlying spaces. Now apply [22], Proposition 2.5.10. □

Let $\mathcal{T} \in \{\mathcal{T}_{\text{ét}}, \mathcal{T}_{\text{Nis}}, \mathcal{T}_{\text{Zar}}, \mathcal{T}_{\text{triv}}\}$ be a topology on $\mathcal{SCR}$. If $X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $Y = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ are two ringed ∞-topoi which are local for the topology $\mathcal{T}$, then we shall write $\mathrm{Hom}_{\mathcal{T}}(X, Y)$ for the ∞-category of $\mathcal{T}$-local morphisms of ringed topoi.

Lemma 4.3.5. *Let $\mathcal{X}$ be an ∞-topos, $\widetilde{\mathcal{O}}$ a $\mathcal{SCR}$-valued presheaf on $\mathcal{X}$, U an object of $\mathcal{X}$, and $\widetilde{A}$ an algebra which is locally of finite presentation over $\widetilde{\mathcal{O}}(U)$. Let $\widetilde{\mathcal{F}}$ be the $\mathcal{S}$-valued presheaf on $\mathcal{X}_{/U}$ defined by*

$$\widetilde{\mathcal{F}}(V) = \mathrm{Hom}_{\widetilde{\mathcal{O}}(U)}(\widetilde{A}, \widetilde{\mathcal{O}}(V)).$$

Let $\mathcal{O}$ denote the sheafification of $\widetilde{\mathcal{O}}$, let $A = \widetilde{A} \otimes_{\widetilde{\mathcal{O}}(U)} \mathcal{O}(U)$, let $\psi : \mathcal{O}(U) \to A$ be the natural morphism, and let $\mathcal{F} = \mathrm{Sol}(\psi)$. Then the natural morphism $\widetilde{\mathcal{F}} \to \mathcal{F}$ identifies $\mathcal{F}$ with the sheafification of $\widetilde{\mathcal{F}}$

Proof. Replacing $\mathcal{X}$ by $\mathcal{X}_{/U}$, we may reduce to the case where U is the final object of $\mathcal{X}$. Let $R = \widetilde{\mathcal{O}}(U)$. Then we may regard $\widetilde{\mathcal{O}}$ as a sheaf of R-algebras. We note that $\mathcal{SCR}_{R/}$ is compactly generated. Since the forgetful functor $\mathcal{SCR}_{R/} \to \mathcal{SCR}$ is right adjoint to $\bullet \otimes_{\mathbf{Z}} R$ and commutes with filtered colimits, we see that the sheafification of $\widetilde{\mathcal{O}}$ as a $\mathcal{SCR}$-valued presheaf agrees with its sheafification as a $\mathcal{SCR}_{R/}$-valued presheaf. Since $\widetilde{A}$ is a compact object of $\mathcal{SCR}_{R/}$, the assertion follows immediately from the description of $\mathcal{SCR}_{R/}$-valued sheaves given in the proof of Proposition 4.1.7. □

Proposition 4.3.6. *Let $\mathcal{T} \in \{\mathcal{T}_{\acute{e}t}, \mathcal{T}_{\mathrm{Nis}}, \mathcal{T}_{\mathrm{Zar}}, \mathcal{T}_{\mathrm{triv}}\}$ be a topology and $n \in \mathbf{Z}_{\geq 0}$.*

1. *A ringed ∞-topos $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is local for $\mathcal{T}$ if and only if $(\mathcal{X}, \tau_{\leq n} \mathcal{O}_{\mathcal{X}})$ is local for $\mathcal{T}$.*

2. *A morphism $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ of ringed ∞-topoi is $\mathcal{T}$-local if and only if the induced morphism $(\mathcal{X}, \tau_{\leq n} \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \tau_{\leq n} \mathcal{O}_{\mathcal{Y}})$ is $\mathcal{T}$-local.*

Proof. We now prove (1). First, suppose that $(\mathcal{X}, \tau_{\leq n} \mathcal{O}_{\mathcal{X}})$ is $\mathcal{T}$-local. Let $\{\psi_\alpha : \mathcal{O}(U) \to A_\alpha\}$ be a family of morphisms and let $\{\psi'_\alpha : (\tau_{\leq n} \mathcal{O})(U) \to A_\alpha \otimes_{\mathcal{O}(U)} (\tau_{\leq n} \mathcal{O})(U)\}$ be the induced family. If ψ_α constitutes an admissible covering of $\mathcal{O}(U)$, then ψ'_α constitutes an admissible covering of $(\tau_{\leq n} \mathcal{O})(U)$, so that the objects $\{\mathrm{Sol}(\psi'_\alpha) \to U\}$ cover the object $U \in \mathcal{X}$. To prove that $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is $\mathcal{T}$-local, it will suffice to prove that $\mathrm{Sol}(\psi'_\alpha) \simeq \mathrm{Sol}(\psi_\alpha)$. By definition, $\mathrm{Sol}(\psi_\alpha)$ represents the sheaf

$$V \mapsto \mathrm{Hom}_{\mathcal{O}(U)}(A, \mathcal{O}(V)).$$

Since A is étale over $\mathcal{O}(U)$, the right hand side may be rewritten as $\mathrm{Hom}_{\tau_{\leq n} \mathcal{O}(U)}(\tau_{\leq n} A, \tau_{\leq n} \mathcal{O}(V))$, and $\tau_{\leq n} A = \tau_{\leq n} \mathcal{O}(U) \otimes_{\mathcal{O}(U)} A$. Lemma 4.3.5 shows that $\mathrm{Sol}(\psi'_\alpha)$ is the sheafification of this presheaf. Since $\mathrm{Sol}(\psi_\alpha)$ is already a sheaf, we get $\mathrm{Sol}(\psi_\alpha) \simeq \mathrm{Sol}(\psi'_\alpha)$.

For the reverse implication, let us suppose that $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is $\mathcal{T}$-local. We must prove that $(\mathcal{X}, \tau_{\leq n} \mathcal{O}_{\mathcal{X}})$ is $\mathcal{T}$-local. Suppose that $U \in \mathcal{X}$, and let $\{\psi'_\alpha : (\tau_{\leq n} \mathcal{O}_{\mathcal{X}})(U) \to A'_\alpha\}$ be an admissible covering family. We must show that the family $\{\mathrm{Sol}(\psi'_\alpha) \to U\}$ is covering. Without loss of generality, we may assume that there are only finitely many elements of the covering. Suppose that we can find an admissible covering $\{\psi''_\alpha : \tau_{\leq n}(\mathcal{O}_{\mathcal{X}} U) \to A''_\alpha\}$ which induces ψ' after base change. Since the categories of étale algebras over $\tau_{\leq n}(\mathcal{O}_{\mathcal{X}}(U))$ and over $\mathcal{O}_{\mathcal{X}}(U)$ are equivalent, ψ''_α is the base change of an admissible morphism $\psi_\alpha : \mathcal{O}_{\mathcal{X}}(U) \to A_\alpha$. Then the family $\{\psi_\alpha\}$ is an admissible covering of $\mathcal{O}_{\mathcal{X}}(U)$, and so the hypothesis implies that $\{\mathrm{Sol}(\psi_\alpha)\}$ forms a covering of U. But the argument for the first part shows that $\mathrm{Sol}(\psi_\alpha) \simeq \mathrm{Sol}(\psi'_\alpha)$, and we are done.

In general, there is no reason to expect that we can find such a family $\{\psi''_\alpha\}$ globally. However, Lemma 4.1.14 (applied to the geometric morphism $\mathcal{X} \to \mathcal{S}^{\mathcal{X}_\kappa^{op}}$ for a large regular cardinal κ) shows that $\{\psi''_\alpha\}$ can always be found locally on U. Since the conclusion is also local on U, the proof of (1) is complete.

The proof of (2) is similar and is left to the reader. □

Remark 4.3.7. Let $(\mathcal{X}, \mathcal{O})$ be a ringed ∞-topos. Then $(\mathcal{X}, \mathcal{O})$ is local for the topology $\mathcal{T}$ if and only if $(\mathcal{X}, \pi_0 \mathcal{O})$ is local for the topology $\mathcal{T}$, if and only if the ordinary ringed topos $(\tau_{\leq 0} \mathcal{X}, \pi_0 \mathcal{O})$ is local for the topology $\mathcal{T}$, in the obvious sense. If the topos $\tau_{\leq 0} \mathcal{X}$ has enough points, then this is equivalent to a condition on stalks: a sheaf of discrete rings $\mathcal{O}$ is local for the topology $\mathcal{T}_{\text{Zar}}$ ($\mathcal{T}_{\text{ét}}$) if and only if for every point x, the stalk $\pi_0(\mathcal{O}_x)$ is a local (strictly Henselian) ring. The situation for the Nisnevich topology is slightly more complicated: if each stalk $\pi_0(\mathcal{O}_x)$ is Henselian, then $\mathcal{O}$ is $\mathcal{T}_{\text{Nis}}$-local; the converse holds provided that each stalk $\mathcal{O}_x$ is Noetherian.

We can now apply the results of the last section to discuss spectra of ringed ∞-topoi. If $(\mathcal{X}, \mathcal{O})$ is a ringed ∞-topos, then we let $\operatorname{Spec}^{\mathcal{T}}(\mathcal{X}, \mathcal{O})$ denote its spectrum with respect to the compactly generated, admissible topology $\mathcal{T}$. If $\mathcal{X}$ is a point and $\mathcal{O} = A \in \mathcal{SCR}$, then we write $\operatorname{Spec}^{\mathcal{T}} A$ instead of $\operatorname{Spec}^{\mathcal{T}}(\mathcal{X}, \mathcal{O})$. If $\mathcal{T}$ is the étale topology, we will simply write $\operatorname{Spec}(\mathcal{X}, \mathcal{O})$ or $\operatorname{Spec} A$.

Remark 4.3.8. The explicit construction of spectra given in the last section shows that if A is an ordinary commutative ring, then the underlying ∞-topos of $\operatorname{Spec}^{\mathcal{T}} A$ is the ∞-topos associated to the ordinary category of set-valued sheaves on A with respect to the topology $\mathcal{T}$.

More generally, suppose that $\mathcal{T}$ and $\mathcal{T}'$ are admissible topologies such that $\mathcal{T}$ is *finer* than $\mathcal{T}'$ in the sense that $\mathcal{T}$ has more admissible morphisms and more covering families; we shall denote this by writing $\mathcal{T} \leq \mathcal{T}'$. In this case, we can define a relative spectrum $\operatorname{Spec}^{\mathcal{T}}_{\mathcal{T}'}(\mathcal{X}, \mathcal{O})$ which is defined for $\mathcal{T}'$-local ringed ∞-topoi $(\mathcal{X}, \mathcal{O})$. By definition, $\operatorname{Spec}^{\mathcal{T}}_{\mathcal{T}'}(\mathcal{X}, \mathcal{O})$ is universal among $\mathcal{T}$-local ringed ∞-topoi which admit a $\mathcal{T}'$-local morphism to $(\mathcal{X}, \mathcal{O})$. To construct $\operatorname{Spec}^{\mathcal{T}}_{\mathcal{T}'}(\mathcal{X}, \mathcal{O})$, one forms a lax fiber product of $\mathcal{X}$ with $\mathcal{Y}_{\mathcal{T}}$ over $\mathcal{Y}_{\mathcal{T}'}$, where $\mathcal{Y}_{\mathcal{T}}$ and $\mathcal{Y}_{\mathcal{T}'}$ denote the classifying ∞-topoi for $\mathcal{T}$-local and $\mathcal{T}'$-local $\mathcal{SCR}$-valued sheaves. We note that $\operatorname{Spec}^{\mathcal{T}}_{\mathcal{T}}$ is the identity functor, and that $\operatorname{Spec}^{\mathcal{T}}_{\mathcal{T}'} \operatorname{Spec}^{\mathcal{T}'}_{\mathcal{T}''} \simeq \operatorname{Spec}^{\mathcal{T}}_{\mathcal{T}''}$ for $\mathcal{T} \leq \mathcal{T}' \leq \mathcal{T}''$.

We now show that the construction of spectra is insensitive to the higher homotopy groups of the structure sheaf.

Proposition 4.3.9. *Suppose that $\mathcal{T}, \mathcal{T}' \in \{\mathcal{T}_{\text{triv}}, \mathcal{T}_{\text{Zar}}, \mathcal{T}_{\text{Nis}}, \mathcal{T}_{\text{ét}}\}$ are topologies with $\mathcal{T} \leq \mathcal{T}'$. Let $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a ringed ∞-topos which is $\mathcal{T}'$-local. If $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = \operatorname{Spec}^{\mathcal{T}}_{\mathcal{T}'}(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$, then $(\mathcal{X}, \tau_{\leq n} \mathcal{O}_{\mathcal{X}}) = \operatorname{Spec}^{\mathcal{T}}_{\mathcal{T}'}(\mathcal{Y}, \tau_{\leq n} \mathcal{O}_{\mathcal{Y}})$.*

Proof. It suffices to treat the universal case in which $\mathcal{Y}$ is the classifying ∞-topos for $\mathcal{T}'$-local $\mathcal{SCR}$-valued sheaves. In this case, $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) = \operatorname{Spec}^{\mathcal{T}'}_{\mathcal{T}_{\text{triv}}}(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$, where $\mathcal{Z}$ is the classifying ∞-topos for arbitrary $\mathcal{SCR}$-valued sheaves. The result follows if we can show that $(\mathcal{Y}, \tau_{\leq n} \mathcal{O}_{\mathcal{Y}}) = \operatorname{Spec}^{\mathcal{T}'}_{\mathcal{T}_{\text{triv}}}(\mathcal{Z}, \tau_{\leq n} \mathcal{O}_{\mathcal{Z}})$ and $(\mathcal{X}, \tau_{\leq n} \mathcal{O}_{\mathcal{X}}) \simeq \operatorname{Spec}^{\mathcal{T}}_{\mathcal{T}_{\text{triv}}}(\mathcal{Z}, \tau_{\leq n} \mathcal{O}_{\mathcal{Z}})$. In other words, we may assume that $\mathcal{T}'$ is the trivial topology, so that $\mathcal{Y}$ is an ∞-category of presheaves. In this case, the result follows immediately from the construction of spectra given in the last section (after noting that the category of $\mathcal{T}$-admissible A-algebras is equivalent to the category of $\mathcal{T}$-admissible $\tau_{\leq n} A$-algebras). □

Remark 4.3.10. We remark that most of the ∞-topoi which we will be considering (for example, the ∞-topos of étale sheaves on a commutative ring) do not necessarily have enough points. This could be remedied by replacing the topologies we are using by the corresponding "hypertopologies". Although this leads to a few simplifications, we feel that our approach is more natural (and equivalent in all practical respects).

Remark 4.3.11. If $A \in \mathcal{SCR}$, then we let A_{red} denote the ordinary commutative ring obtained from $\pi_0 A$ by dividing out the nilradical. All of the assertions made in this section regarding the functors $A \mapsto \tau_{\leq n} A$ may be modified to include also the case $A \mapsto A_{\text{red}}$, since the category of étale A_{red}-algebras is equivalent to the category of étale $\pi_0 A$-algebras.

Remark 4.3.12. If $A \in \mathcal{SCR}$, then A is naturally equivalent to the global sections of the structure sheaf on $\operatorname{Spec}^{\mathcal{T}} A$. This does not follow formally from the construction of spectra. If we understand the underlying ∞-topos of $\operatorname{Spec}^{\mathcal{T}} A$ to be the ∞-category of sheaves on the category of étale A-algebras, then the structure sheaf is given by sheafifying the presheaf given by the forgetful functor

$$\mathcal{SCR}^{\text{ét}}_{A/} \rightarrow \mathcal{SCR}.$$

We observe that this presheaf is already a sheaf: in other words, each of the topologies that we are considering is *sub-canonical*. Even more, the flat hypertopology is subcanonical: see Example 4.2.4.

4.4 Finiteness Properties

The goal of this section is to formulate some finiteness properties enjoyed by the topologies $\mathcal{T}_{\text{Nis}}$, $\mathcal{T}_{\text{Zar}}$, and $\mathcal{T}_{\text{ét}}$. The results of this section deal purely with "topological" properties of algebro-geometric objects, so there is no need to consider simplicial commutative rings: all rings in this section are assumed to be discrete. Also, throughout this section we shall write $\operatorname{Spec} A$ for the Zariski spectrum of a commutative ring A, regarded as a topological space.

Most of the results of this section concern the Nisnevich topology, and will not be needed later in this paper. This section may be safely skipped, with the exceptions of the second parts of Theorems 4.4.3 and 4.4.4.

We first justify the assertion made in Remark 4.2.2, regarding the relationship between our definition of the Nisnevich topology and the usual definition.

Proposition 4.4.1. *Let A be a Noetherian commutative ring. Then a family $\{A \rightarrow A_\alpha\}$ of étale A-algebras is covering with respect to the Nisnevich topology if and only if, for any residue field κ of A, there exists an index α such that $A_\alpha \otimes_A \kappa$ contains a factor isomorphic to κ.*

Proof. It is clear that any Nisnevich covering has the indicated property (this does not require the assumption that A is Noetherian). For the converse, let us suppose that we are given a family of étale maps $\{A \rightarrow A_\alpha\}$ which satisfy the hypotheses of the proposition. We define a sequence of subsets U_i of the Zariski spectrum X of A as follows. Let $U_0 = \emptyset$. Assuming that $U_i \neq X$ has been defined, let η be a generic point of the closed subset $X - U_i$, let κ be the residue field of A at η, and let A_α be such that $A_\alpha \times_A \kappa$ contains a factor isomorphic to κ. Then $\operatorname{Spec} A_i \times_X (X - U_i)$ is étale over $X - U_i$ and contains an open subset which maps to $X - U_i$ via an open immersion. We let U_{i+1} denote the union of U with the image of this open immersion. Since A is Noetherian, the sequence of open sets $U_0 \subset U_1 \ldots$ cannot be continued indefinitely. Thus we eventually have $U_n = X$, and the construction shows that $\{A \to A_\alpha\}$ is a Nisnevich covering of A. □

Let A be a commutative ring and $\mathcal{F}$ a $\mathcal{S}$-valued presheaf on the category of étale A-algebras which is a sheaf for the Nisnevich topology. In particular, for any étale A-algebra A', $\mathcal{F}$ restricts to a Zariski sheaf on $\operatorname{Spec} A'$. Consequently, $\mathcal{F}$ extends uniquely to a Zariski sheaf on the category of quasi-compact, quasi-separated étale A-schemes. We begin by stating an appropriate version of the Morel-Voevodsky descent theorem.

Proposition 4.4.2. *Let R be a commutative ring, and let $\mathcal{F}$ be a $\mathcal{S}$-valued presheaf on the category $\mathcal{C}$ of schemes which are quasi-compact, quasi-separated, and étale over R. Then $\mathcal{F}$ is a sheaf for the Nisnevich topology if and only if the following condition is satisfied:*

- *For any $X \in \mathcal{C}$, any quasi-compact open $U \subseteq X$, and any $\pi : X' \to X$ in $\mathcal{C}$, if π is an isomorphism over $X - U$, then $\mathcal{F}(X) = \mathcal{F}(U) \times_{\mathcal{F}(U')} \mathcal{F}(X')$, where $U' = X' \times_X U$.*

This result is usually stated for a slightly different topology than the one which we consider: what one might call the "Nisnevich hypertopology", which imposes descent conditions for arbitrary hypercoverings. In this setting, one must make the assumption that R is Noetherian and of finite Krull dimension. As we shall see in a moment, this implies that the Nisnevich topology has finite homotopy dimension so that *any* Nisnevich sheaf actually satisfies this stronger descent condition. On the other hand, Proposition 4.4.2 is valid for arbitrary rings R if one requires only Čech descent and employs our definition of the Nisnevich topology.

We next apply Proposition 4.4.2 to show that sheafification with respect to the Nisnevich topology commutes with filtered colimits:

Theorem 4.4.3. *1. Let R be a commutative ring, let $\mathcal{X}$ denote the ∞-topos of $\mathcal{S}$-valued sheaves on the Nisnevich (Zariski) topology of R. Let U be a quasi-compact, quasi-separated scheme which is étale over R. Then evaluation on U gives a functor*

$$\mathcal{X} \to \mathcal{S}$$

which commutes with filtered colimits. (In other words, U is a compact object of $\mathcal{X}$.)

2. Let R be a commutative ring, let $\mathcal{X}$ denote the ∞-topos of $\mathcal{S}$-valued sheaves on the étale topology of R. Let U be a quasi-compact, quasi-separated scheme which is étale over R. Then, for each integer $k \geq 0$, evaluation on U gives a functor

$$\tau_{\leq k} \mathcal{X} \to \mathcal{S}$$

which commutes with filtered colimits. (In other words, U is a compact object of $\tau_{\leq k}\,\mathcal{X}$.)

Proof. We first give the proof of (1) for the Nisnevich topology (the proof for the Zariski topology is similar but easier and left to the reader). Let $\{\mathcal{F}_\alpha\}$ be a filtered diagram of objects of $\mathcal{X}$. Let $\mathcal{F}$ be defined on quasi-compact, quasi-separated schemes which are étale over R by the formula

$$\mathcal{F}(U) = \operatorname{colim}_\alpha \mathcal{F}_\alpha(U).$$

Using Proposition 4.4.2, it follows that $\mathcal{F}$ is a sheaf for the Nisnevich topology. Clearly $\mathcal{F}$ is the filtered colimit of the diagram $\{\mathcal{F}_\alpha\}$ in $\mathcal{X}$.

The proof of (2) is similar, but we must work a little bit harder to show that $\mathcal{F}$ is a sheaf. It will suffice to show that $\mathcal{F}$ carries coproducts into products and that if $U_0 \to V$ is an étale surjection, then the natural map $\mathcal{F}(V) \to |\,\mathcal{F}(U_\bullet)|$ is an equivalence, where U_n denotes the $(n+1)$-fold fiber power of U_0 over V. The first claim follows easily from the assumption that each $\mathcal{F}_\alpha$ carries coproducts into products. For the second, we must be more careful since the formation of the geometric realization of a cosimplicial object is not a finite limit and does not commute with filtered colimits in general. However, if each $\mathcal{F}_\alpha$ is k-truncated, then $\mathcal{F}$ is also k-truncated, and consequently $|\,\mathcal{F}(U_\bullet)|$ and $|\,\mathcal{F}_\alpha(U_\bullet)|$ are equivalent to finite limits which mention only the k-skeleton of the simplicial scheme $X_\bullet$. The conclusion then follows from the fact that finite limits distribute over filtered colimits. □

Proposition 4.4.2 also implies that in some sense, the Nisnevich spectrum of a ring R may be constructed as the limit of the Nisnevich topologies of finitely generated subalgebras of R. More specifically, we have the following:

Theorem 4.4.4. *1. Let A be a commutative ring, and let $\mathcal{F}$ be a $\mathcal{S}$-valued sheaf on the Nisnevich (Zariski) topology of A. Extend the definition of $\mathcal{F}$ to all A-algebras B by setting*

$$\mathcal{F}(B) = (\pi^*\,\mathcal{F})(B)$$

where π is the geometric morphism from the Nisnevich (Zariski) spectrum of B to the Nisnevich (Zariski) spectrum of A. Then the functor $\mathcal{F}$ commutes with filtered colimits.

2. Let A be a commutative ring, and let $\mathcal{F}$ be a truncated $\mathcal{S}$-valued sheaf on the étale topology of A. Extend the definition of $\mathcal{F}$ to all A-algebras B by setting

$$\mathcal{F}(B) = (\pi^*\,\mathcal{F})(B)$$

where π is the geometric morphism from the étale spectrum of B to the étale spectrum of A. Then the functor $\mathcal{F}$ commutes with filtered colimits.

Proof. We first prove (1) for the Zariski topology. Let $\mathcal{F}'$ be the $\mathcal{S}$-valued functor on A-algebras which agrees with $\mathcal{F}'$ on finitely presented A-algebras and commutes with filtered colimits. If R is an arbitrary A-algebra, then $\mathcal{F}'$ restricts to a $\mathcal{S}$-valued presheaf on the category $\mathcal{C}$ of admissible R-algebras. It is easy to see that $\mathcal{F}\,|\,\mathcal{C}$ is the sheafification of $\mathcal{F}'\,|\,\mathcal{C}$ with respect to the Zariski topology. To complete the proof, it will suffice to show that $\mathcal{F}'$ is already a sheaf with respect to the Zariski topology. In other words, we must show that for any admissible R-algebra R' and any Zariski covering $\{R' \to R'_\alpha\}$, $\mathcal{F}(R')$ may be computed as the homotopy limit of an appropriate diagram. Without loss of generality, we may refine the covering and thereby assume that it is indexed by a finite set. Next, since each R'_α is finitely presented as an R'-algebra, we may assume that they are all defined over some finitely presented A-algebra. Enlarging A if necessary, we may assume that $R'_\alpha = R' \times_A A_\alpha$, where $\{A \to A_\alpha\}$ is a Zariski covering. Now the desired result follows from the fact that the diagram defining the appropriate limit is *finite*, and finite limits commute with filtered colimits.

Next we prove (1) for the Nisnevich topology. Since $\mathcal{F}'$ is a sheaf for the Zariski topology, it admits a unique extension (as a Zariski sheaf) to the category of all quasi-compact, quasi-separated A-schemes. We now run the above argument again, this time using Proposition 4.4.2, to deduce that $\mathcal{F}'$ is a sheaf for the Nisnevich topology and complete the proof.

The proof of (2) is similar, except that we verify that $\mathcal{F}'$ is an étale sheaf more directly. Since it is clear that $\mathcal{F}'$ carries products into products, it suffices to show that if $R \to R_0$ is a faithfully flat, étale map, and R_i denotes the $(i+1)$-fold tensor power of R_0 over R, then $\mathcal{F}'(R) \to |\mathcal{F}'(R_\bullet)|$ is an equivalence. Since R_0 is finitely presented over R, it is the base change of an R_α-algebra $(R_\alpha)_0$ for some map $R_\alpha \to R$, where R_α is finitely presented over A. Enlarging α if necessary, we may suppose that $(R_\alpha)_0$ is faithfully flat and étale over R_α. We may now attempt to deduce that $\mathcal{F}'(R) \simeq |\mathcal{F}'(R_\bullet)|$ by knowing that this holds over R_β cofinally, and passing to filtered colimits. The situation is as in the proof of Theorem 4.4.3. In general, filtered colimits do not commute with the geometric realization of cosimplicial spaces. However, they do commute in the special case where all of the spaces are k-truncated, since in this case the geometric realization is equivalent to a finite limit. □

Theorem 4.4.5. *Let R be a Noetherian commutative ring of Krull dimension $\leq n$. Let $\mathcal{X}$ denote the underlying ∞-topos of $\operatorname{Spec}^{\mathcal{T}_{\mathrm{Nis}}} R$. Then $\mathcal{X}$ has homotopy dimension $\leq n$.*

(For the definition of homotopy dimension, we refer the reader to §4 of [22].)

Proof. We give an argument which is modelled on the version of the Grothendieck's vanishing theorem proven in [22]. Some modifications are necessary because the Nisnevich topos is not localic, but on the whole the argument becomes a bit simpler because one does not need to worry about noncompact open sets.

If X is a scheme admitting a quasi-finite map to $\operatorname{Spec} R$, we define the *ambient dimension* of X to the Krull dimension of the closure of the image of X in $\operatorname{Spec} A$. We note that if $U \subseteq X$ is a dense open subset and U has ambient dimension $\leq k$, then $X - U$ has ambient dimension $< k$.

Let X be a scheme which is étale over $\operatorname{Spec} R$, and let $\mathcal{F}$ be a $\mathcal{S}$-valued sheaf on the Nisnevich topology of X. We shall say that $\mathcal{F}$ is *strongly k-connected* if the following condition is satisfied: for any scheme X' which is étale over X, any $m \geq -1$, and any map $\phi : S^m \to \mathcal{F}(X')$, there exists an open subset $U \subseteq X'$ such that $X' - U$ has ambient dimension $< m - k$, a Nisnevich covering $X'' \to U$, and a nullhomotopy of $\phi | X''$.

If R has Krull dimension $\leq n$, then any $(k+n)$-connected $\mathcal{S}$-valued sheaf $\mathcal{F}$ is strongly k-connected. To prove the theorem, it will suffice to prove that if $\mathcal{F}$ is strongly (-1)-connected sheaf on X, then $\mathcal{F}(X)$ is nonempty. In order for the proof to go through, we will need to prove the following slightly stronger statement:

- Let X be any scheme which is étale and of finite type over $\operatorname{Spec} R$, let $\mathcal{F}$ be any strongly $(-k)$-connected $\mathcal{S}$-valued sheaf on the Nisnevich topology of X, let $U \subseteq X$ be any open set, and let $\eta \in \mathcal{F}(U)$. Then there exists an open set V containing U and an extension of η to V, such that $X - V$ has ambient dimension $< k - 1$.

The proof goes by descending induction on k. For the base case we may take $k = n + 2$ and $V = U$ (since *any* subset of X has ambient dimension $\leq n$).

Now suppose that the result has been established for strongly $(-k-1)$-connected sheaves (on arbitrary étale $\operatorname{Spec} R$-schemes), and let X, U, $\mathcal{F}$, and η be as above where $\mathcal{F}$ is strongly $(-k)$-connected. Consider all open sets $\widetilde{U} \subseteq X$ such that $\widetilde{U}$ contains U and η extends over $\widetilde{U}$. Since X is a Noetherian topological space, we may choose $\widetilde{U}$ to be maximal with respect to these properties. Replacing U by $\widetilde{U}$, we may reduce to the case where U is itself maximal.

Since $\mathcal{F}$ is strongly $(-k)$-connected, there exists an open set $W \subseteq X$ and a Nisnevich covering $X' \to W$ such that $\mathcal{F}$ admits a global ζ section over X', and $X - W$ has ambient dimension $< k - 1$. Without loss of generality, we may assume that W contains U, and we may replace X by W and thereby assume that $\mathcal{F}$ is (-1)-connected.

Let $\mathcal{F}'$ denote the Nisnevich $\mathcal{S}$-valued sheaf on $U' = X' \times_X U$ consisting of paths from $\zeta|V'$ to $\eta|V'$. Then $\mathcal{F}'$ is strongly $(-k-1)$-connected, so by the inductive hypothesis there exists a closed subset $K' \subseteq U'$ having ambient dimension $< k$ such that $\zeta|(U' - K') \simeq \eta|(U' - K')$. Then $\overline{K'} - K'$ has ambient dimension $< k-1$. Removing the closure of the image of $\overline{K'} - K'$ from X, we may suppose that K' is closed in X'.

We now claim that $U = X$. If not, let x denote a generic point of some component of $X - U$. There exists a point $x' \in X' - U'$ such that the projection $p : X' - U' \to X - U$ is a local homeomorphism at x'. Let V be a neighborhood of x' in $X' - U'$ such that $p|V$ is an open immersion. Then the pair $\{U, U' \cup V\}$ constitutes a Nisnevich covering of $U \cup p(V)$. By construction, the sections η and $\zeta|(U' \cup V)$ may be glued along the overlap U' to give a section of $\mathcal{F}$ over $U \cup p(V)$ which extends η. By the maximality of U, we get $U \cup p(V) = U$, so that $x \in U$, which is a contradiction. $\square$

Remark 4.4.6. The additional hypotheses of truncatedness given in Propositions 4.4.3 and 4.4.4 are necessary when working with the étale topology. For example, if $\mathbf{R}$ is the field of real numbers, then the étale spectrum of $\mathbf{R}$ is the classifying ∞-topos for the Galois group $\operatorname{Gal}(\mathbf{C}/\mathbf{R}) \simeq \mathbf{Z}/2\mathbf{Z}$. The classifying space of $\mathbf{Z}/2\mathbf{Z}$ is not homotopy equivalent to a finite complex, so that the global section functor does not commute with filtered colimits. However, the classifying space of any finite group does admit a CW decomposition with only finitely many cells in each dimension, so that the functor of global sections commutes with filtered colimits when restricted to k-truncated sheaves of spaces for any $k \geq 0$.

One of the advantages of the Nisnevich topology is that it is coarse enough to have the good finiteness properties established in this section (unlike the étale topology), yet fine enough to allow the proof of Artin's representability theorem to go through (unlike the Zariski topology).

4.5 Derived Schemes

In this section we return to our usual convention regarding the definition of $\operatorname{Spec} A$: unless otherwise specified, the spectrum is taken with respect to the étale topology.

We are now finally in a position to give our main definition.

Definition 4.5.1. A *derived scheme* is a ringed ∞-topos $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ with the following property: there exists a collection of objects $U_\alpha \in \mathcal{X}$ such that $\coprod_\alpha U_\alpha \to 1_{\mathcal{X}}$ is surjective, and each $(\mathcal{X}_{/U_\alpha}, \mathcal{O}_{\mathcal{X}} | U_\alpha)$ is equivalent to $\operatorname{Spec} A_\alpha$ for some $A_\alpha \in \mathcal{SCR}$.

Remark 4.5.2. Replacing the étale topology by some other admissible topology $\mathcal{T}$ on $\mathcal{SCR}$, we would arrive at the notion of a $\mathcal{T}$-derived scheme. For example, when $\mathcal{T}$ is the trivial topology, a $\mathcal{T}$-derived scheme is simply a ringed ∞-topos $(\mathcal{X}, \mathcal{O})$ such that $\mathcal{X}$ is étale over the ∞-topos $\mathcal{S}$.

Let $\mathcal{T} \leq \mathcal{T}'$, and suppose that $(\mathcal{X}, \mathcal{O})$ is a $\mathcal{T}'$-derived scheme. Then $\operatorname{Spec}^{\mathcal{T}}_{\mathcal{T}'}(\mathcal{X}, \mathcal{O})$ is a $\mathcal{T}$-derived scheme. This follows immediately from the locality and transitivity properties of spectrification.

If $\mathcal{X}$ and $\mathcal{Y}$ are derived schemes, then we define the ∞-category of derived scheme morphisms from $\mathcal{X}$ to $\mathcal{Y}$ to be ∞-category $\operatorname{Hom}_{\mathcal{T}_{\text{ét}}}(\mathcal{X}, \mathcal{Y})$ of $\mathcal{T}_{\text{ét}}$-local morphisms of ringed ∞-topoi. We will show in §4.6 that this ∞-category is a small ∞-groupoid, which we may identify with its classifying space. In other words, the derived schemes constitute an ordinary ∞-category. At this point, we know only that they form an $(\infty, 2)$-category (potentially with "large" morphism ∞-categories).

Remark 4.5.3. If $(\mathcal{X}, \mathcal{O})$ is a ringed ∞-topos whose underlying ∞-topos $\mathcal{X}$ is a disjoint union of components $\mathcal{X}_\alpha$, then $(\mathcal{X}, \mathcal{O})$ is a derived scheme if and only if each $(\mathcal{X}_\alpha, \mathcal{O} \,|\, \mathcal{X}_\alpha)$ is a derived scheme.

Remark 4.5.4. The property of being a derived scheme is *local*. That is, if $(\mathcal{X}, \mathcal{O})$ is a derived scheme and $f : \mathcal{Y} \to \mathcal{X}$ is an étale geometric morphism, then $(\mathcal{Y}, f^* \mathcal{O})$ is a derived scheme. Conversely, if f is an étale surjection and $(\mathcal{Y}, f^* \mathcal{O})$ is a derived scheme, then $(\mathcal{X}, \mathcal{O})$ is a derived scheme.

Remark 4.5.5. Let $f : X \to Y$ is a morphism of derived schemes. Locally on X and Y, we may write $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$; then specifying f is equivalent to specifying a morphism $B \to A$ in $\mathcal{SCR}$. If P is any property of $\mathcal{SCR}$-morphisms which is local on both the source and the target with respect to the étale topology, then it makes sense to say that $f : X \to Y$ has the property P if all of the $\mathcal{SCR}$-morphisms $B \to A$ which are locally associated to f have the property P. In particular, we may speak of morphisms of derived schemes being smooth, flat, locally of finite presentation, almost of finite presentation, étale and so forth. We note that $f : X \to Y$ is étale in the sense just described if and only if it is étale as a map of $\mathcal{SCR}$-structured ∞-topoi.

Example 4.5.6. A ringed ∞-topos $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a $\mathcal{T}_{\text{triv}}$-derived scheme if and only if $\mathcal{X} \simeq \mathcal{S}_{/E}$ for some $E \in \mathcal{S}$.

Our main goal in this section is to compare our notion of a derived scheme with the more classical notion of a scheme. Roughly speaking, our notion is more general in two essentially different ways: we consider ringed ∞-topoi $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ in which the $\mathcal{O}_{\mathcal{X}}$ is not discrete, and in which $\mathcal{X}$ is not necessarily associated to a topological space (or even a topos).

Proposition 4.5.7. *If $(\mathcal{X}, \mathcal{O})$ is a derived scheme, then $(\mathcal{X}, \tau_{\leq k} \mathcal{O})$ is also a derived scheme.*

Proof. This follows immediately from the last assertion of Proposition 4.3.9. ☐

We next turn our attention to the underlying ∞-topos of a derived scheme. We recall from [22] that if $\mathcal{X}$ is an ∞-topos, then the full subcategory $\tau_{\leq 0} \mathcal{X}$ consisting of discrete objects is an ordinary (Grothendieck) topos. In the reverse direction, for any topos $\mathfrak{X}$, one may construct an ∞-topos $\Delta\mathfrak{X}$. Moreover, these constructions are adjoint to one another so that there is a natural geometric morphism $\mathcal{X} \to \Delta(\tau_{\leq 0} \mathcal{X})$ for any ∞-topos $\mathcal{X}$. The ∞-topoi $\mathcal{X}$ and $\Delta(\tau_{\leq 0} \mathcal{X})$ have the same discrete objects, but not necessarily the same n-truncated objects for $n > 0$. However, we can say the following:

Lemma 4.5.8. *Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi and $n \geq 0$. Suppose that π^* induces an equivalence $\tau_{\leq n} \mathcal{Y} \to \tau_{\leq n} \mathcal{X}$, and that $\tau_{\leq n} \mathcal{Y}$ generates $\mathcal{Y}$ under colimits. Then π^* is fully faithful on $\tau_{\leq n+1} \mathcal{Y}$.*

Proof. Let $A, B \in \mathcal{Y}$. We will show that $\phi : \text{Hom}_{\mathcal{Y}}(A, B) \to \text{Hom}_{\mathcal{X}}(\pi^* A, \pi^* B)$ is an equivalence whenever B is $(n+1)$-truncated. Since $\mathcal{Y}$ is generated by n-truncated objects, we may assume that A is n-truncated. Given two morphisms $\alpha, \beta : A \to B$, the space of paths from α to β is n-truncated and compatible with the functor π^*. The assumption then implies that the space of paths from α to β is equivalent to the space of paths from $\pi^* \alpha$ to $\pi^* \beta$. In other words, ϕ is an inclusion; to show that ϕ is an equivalence we need only show that it is surjective on π_0: that is, any map $\pi^* A \to \pi^* B$ is induced by a morphism $A \to B$, up to homotopy. Let $\widetilde{f} : \pi^* A \to \pi^* B$ be such a morphism.

Choose a surjection $B_0 \to B$ where B_0 is n-truncated. Since B is $(n+1)$-truncated, the induced morphism $\pi^* B_0 \to \pi^* B$ is n-truncated, so that $\widetilde{A}_0 = \pi^* A \times_{\pi^* B} \pi^* B_0$ is n-truncated. Let B_n denote the $(n+1)$-fold product of B_0 with itself over B and $\widetilde{A}_0$ the n-fold product of $\widetilde{A}_0$ with itself over $\pi^* A$. Then $B_\bullet$ is a simplicial resolution of B, and $\widetilde{A}_\bullet$ is the induced simplicial resolution of $\pi^* A$. Consequently, there is a map of simplicial resolutions $\widetilde{A}_\bullet \to \pi^* B_\bullet$. Since all of the objects involved in these resolutions are n-truncated, the hypothesis gives a simplicial resolution $A_\bullet$ of A and a map of resolutions $f_\bullet : A_\bullet \to B_\bullet$ which pulls back to the map $\widetilde{A}_\bullet \to \pi^* B_\bullet$. Let f be the induced map between the simplicial resolutions; then it is clear that $\pi^* f \simeq \widetilde{f}$. ☐

Lemma 4.5.9. *Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism between ∞-topoi. Suppose that π^* induces an equivalence $\tau_{\leq n}\mathcal{Y} \to \tau_{\leq n}\mathcal{X}$ and that $\mathcal{Y}$ is generated by n-truncated objects. Let E be any object of $\mathcal{X}$, and let $\pi' : \mathcal{X}_{/E} \to \mathcal{Y}_{/\pi_* E}$ denote the induced geometric morphism. Then $(\pi')^*$ is fully faithful on $\tau_{\leq n}\mathcal{Y}_{/\pi_* E}$.*

Proof. Since the ∞-category of n-truncated objects over E is equivalent to the ∞-category of n-truncated objects over $\tau_{\leq n+1}E$, we may suppose that E is $(n+1)$-truncated. In this case, any n-truncated object over E is itself $(n+1)$-truncated. Lemma 4.5.8 implies that π^* exhibits $\tau_{\leq n+1}\mathcal{X}$ as a co-localization of $\tau_{\leq n+1}\mathcal{Y}$. In particular, the adjunction morphism $\pi^*\pi_* E \to E$ is an inclusion, and π' induces an identification of $\tau_{\leq n}\mathcal{Y}_{/\pi_* E}$ with the full subcategory of $\tau_{\leq n}\mathcal{X}_{/E}$ consisting of those n-truncated morphisms $A \to E$ which factor through $\pi^*\pi_* E$. □

We will apply these lemmas in the case $n = 0$ to prove the following:

Theorem 4.5.10. *Let $(\mathcal{X}, \mathcal{O})$ be a derived scheme. Then the adjunction morphism $\pi : \mathcal{X} \to \Delta(\tau_{\leq 0}\mathcal{X})$ is étale.*

Proof. Since the assertion is purely topological, we may assume without loss of generality that the structure sheaf $\mathcal{O}$ is discrete. Let $\mathcal{Y}$ denote the ∞-topos $\Delta(\tau_{\leq 0}\mathcal{X})$. The assertion that π is étale is local on $\mathcal{X}$ (where we regard $\mathcal{Y}$ as *fixed*), so it suffices to show that the induced morphism $\mathcal{X}_{/E} \to \mathcal{Y}$ is étale, where the object $E \in \mathcal{X}$ has been chosen so that $(\mathcal{X}_{/E}, \mathcal{O}|E) \simeq \operatorname{Spec} A$ for $A = \mathcal{O}(E)$. Let $E' = \pi_*(\tau_{\leq 1}E)$; then π induces a morphism $\pi' : \mathcal{X}_{/E} \to \mathcal{Y}_{/E'}$ and it will suffice to show that π' is an equivalence.

Since E' is 1-truncated, we may regard it as a sheaf of (ordinary) groupoids on the underlying topos $\tau_{\leq 0}\mathcal{Y} = \tau_{\leq 0}\mathcal{X}$. Then $\mathcal{Y}_{/E'}$ is equivalent to the ∞-topos of sheaves on the topos of discrete objects of $\mathcal{Y}_{/E'}$ (which are the *representations* of the groupoid E' in the topos $\tau_{\leq 0}\mathcal{Y}$). By hypothesis, $\mathcal{X}_{/E}$ is also the ∞-topos associated to its underlying topos of discrete objects (see Remark 4.1.19). Thus, to prove that π' is an equivalence, it will suffice to show that $(\pi')^*$ induces an equivalence between $\tau_{\leq 0}\mathcal{Y}_{/E'}$ and $\tau_{\leq 0}\mathcal{X}_{/E}$.

Since $\tau_{\leq 0}\mathcal{X}_{/E} \simeq \tau_{\leq 0}\mathcal{X}_{/\tau_{\leq 1}E}$, Lemma 4.5.9 implies that $(\pi')^*$ is fully faithful when restricted to discrete objects. To complete the proof, it will suffice to show that $(\pi')^*$ is essentially surjective. Since $(\pi')^*$ commutes with all colimits, it will suffice to show that $\mathcal{X}_{/E}$ is generated by objects lying in the essential image of $(\pi')^*|\tau_{\leq 0}\mathcal{Y}_{/E'}$. We note from the construction of spectra that $\mathcal{X}_{/E} = \operatorname{Spec} A$ is generated by objects of the form $\operatorname{Spec} A'$, where A' is an étale A-algebra.

Now the really essential point is to notice that $\mathcal{O}$ is a *discrete* object of $\mathcal{X}$, so that $\mathcal{O}$ is the pullback of a sheaf of rings on $\mathcal{Y}$ which, to avoid confusion, we shall denote by $\mathcal{O}_{\mathcal{Y}}$. In particular, $\mathcal{O}|E = (\pi')^*\mathcal{O}_{\mathcal{Y}}|E'$. Now we may apply Lemma 4.5.9 again, to the sheaf $\mathcal{O}_{\mathcal{Y}}|E'$ and the final object, to deduce that the ring $A = \mathcal{O}(E)$ is canonically isomorphic with the ring $\mathcal{O}_{\mathcal{Y}}(E')$. In particular, we may view A' as an admissible $\mathcal{O}_{\mathcal{Y}}(E')$-algebra. Let $\psi : \mathcal{O}_{\mathcal{Y}}(E') \to A'$ and $\psi' : \mathcal{X}_{/E} \to A'$ denote the corresponding maps; to complete the proof it will suffice to show that the natural map $(\pi')^*\operatorname{Sol}(\psi) \to \operatorname{Sol}(\psi') = \operatorname{Spec} A'$ is an equivalence. In other words, we need to show that π' is $\mathcal{T}$-local. This follows from Proposition 4.3.6 since $\mathcal{O} = \pi^*\mathcal{O}_{\mathcal{Y}}$. □

Remark 4.5.11. Theorem 4.5.10 has an analogue for orbifolds which is quite a bit easier to prove. Let us define a *higher orbifold* to be an ∞-topos equipped with a sheaf of discrete commutative rings which looks locally like a smooth manifold together with its sheaf of smooth functions. Using the fact that any such ∞-topos is locally equivalent to a localic ∞-topos, one can deduce that any higher orbifold admits a (1-connected) étale morphism to an ordinary orbifold (that is, a higher orbifold whose underlying ∞-topos is equivalent to an ∞-category of sheaves on some topos). In this sense, there is not much benefit in discussing higher orbifolds because they may be understood as ordinary orbifolds together with extra structure. Theorem 4.5.10 implies that the same thing happens with derived schemes, at least when the structure sheaf is discrete.

Now suppose that $(\mathcal{X}, \mathcal{O})$ is a derived scheme whose structure sheaf $\mathcal{O}$ is 0-truncated. In this case, $\mathcal{O}$ is a sheaf of discrete commutative rings on $\mathcal{X}$, which may be thought of as a ring object in the ordinary category $\tau_{\leq 0}\mathcal{X}$. Let $\pi : \mathcal{X} \to \Delta(\tau_{\leq 0}\mathcal{X})$. Then π is an étale surjection, so that $(\Delta(\tau_{\leq 0}\mathcal{X}), \pi_* \mathcal{O})$ is a derived scheme and the derived scheme structure on $\mathcal{X}$ is obtained by pulling back the derived scheme structure on $\Delta(\tau_{\leq 0}\mathcal{X})$. We have proved:

Theorem 4.5.12. *Let $(\mathcal{X}, \mathcal{O})$ be a 0-truncated derived scheme. Then there exists a ringed topos $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ and a 1-connected sheaf of spaces E on $\mathfrak{X}$ such that $(\mathcal{X}, \mathcal{O})$ is equivalent to $((\Delta\mathfrak{X})_{/E}, \mathcal{O}_{\mathfrak{X}}|(\Delta\mathfrak{X})_{/E})$.*

Under the hypotheses of Theorem 4.5.12, the topos $\mathfrak{X}$, the sheaf of rings $\mathcal{O}_{\mathfrak{X}}$ and the object $E \in \Delta\mathfrak{X}$ are canonically determined by $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. Of course, the ringed topos $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is not arbitrary: in order for $(\Delta\mathfrak{X}_{/E}, \mathcal{O}_{\mathfrak{X}})$ to be a derived scheme, the ringed topos $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ must be locally equivalent to the étale spectrum of a commutative ring. This is essentially equivalent to the classical definition of a Deligne-Mumford stack. In other words, up to a 1-connected étale morphism, a 0-truncated derived scheme is just a Deligne-Mumford stack. One recovers exactly the Deligne-Mumford stacks by restricting attention to the case where the 1-connected object E is final.

Remark 4.5.13. For the purposes of this paper, a Deligne-Mumford stack is a topos equipped with a sheaf of (discrete) rings which is locally equivalent to the étale spectrum of a (discrete) commutative ring. This is slightly more general than many standard definitions, which allow only Deligne-Mumford stacks satisfying certain technical hypotheses regarding the diagonal. It is the more general definition that compares well with our notion of a derived scheme.

Remark 4.5.14. The same arguments show that to specify a scheme is equivalent to specifying a $\mathcal{T}_{\mathrm{Zar}}$-derived scheme $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ for which $\mathcal{O}_{\mathcal{X}}$ is 0-truncated and $\mathcal{X}$ is *localic*; that is, $\mathcal{X}$ is generated by its (-1)-truncated objects. The inclusion of the category of ordinary schemes into the 2-category of Deligne-Mumford stacks may be implemented by the relative spectrum functor $\mathrm{Spec}^{\mathcal{T}_{\text{ét}}}_{\mathcal{T}_{\mathrm{Zar}}}$.

Now that we understand derived schemes when the structure sheaf is discrete, let us proceed to give a characterization in the general case. If $(\mathcal{X}, \mathcal{O})$ is any ringed ∞-topos and $i \geq 0$, then the assignment $U \mapsto \pi_i \mathcal{O}(U)$ defines a presheaf of abelian groups on $\mathcal{X}$. We let $\pi_i \mathcal{O}$ denote its sheafification; it is an abelian group object of $\tau_{\leq 0} \mathcal{X}$. In particular, $\pi_0 \mathcal{O} = \tau_{\leq 0} \mathcal{O}$ is a sheaf of commutative rings. We note that each $\pi_i \mathcal{O}$ has the structure of a sheaf of modules over $\pi_0 \mathcal{O}$.

We need to recall a bit of terminology from [22]. If $\mathcal{X}$ is an ∞-topos, and $f : X \to Y$ is a morphism in $\mathcal{X}$, then one can define homotopy groups of the mapping fiber of f as certain sheaves of pointed sets on $\mathcal{X}_{/X}$. The morphism f is said to be ∞-*connected* if these homotopy sheaves all vanish. In many cases, this implies that f is an equivalence; however, this might not be the case for many ∞-topoi of interest to us (such as the étale-spectra of commutative rings). However, it is always possible to modify $\mathcal{X}$ so as to solve this problem by replacing $\mathcal{X}$ by the ∞-category $\mathcal{X}^{\text{hyp}}$ of *hypersheaves* in $\mathcal{X}$. An object $Z \in \mathcal{X}$ is a hypersheaf if it is local with respect to ∞-connected morphisms: that is, any ∞-connected morphism $f : X \to Y$ induces an equivalence $\text{Hom}_{\mathcal{X}}(Y, Z) \to \text{Hom}_{\mathcal{X}}(X, Z)$. We shall say that a $\mathcal{SCR}$-valued sheaf $\mathcal{O}$ on $\mathcal{X}$ is a *hypersheaf* if its underlying sheaf of spaces is a hypersheaf.

Example 4.5.15. If $\mathcal{O}$ is an n-truncated $\mathcal{SCR}$-valued sheaf on $\mathcal{X}$, then $\mathcal{O}$ is a hypersheaf.

We are now in a position to give a characterization of derived schemes.

Theorem 4.5.16. *Let $(\mathcal{X}, \mathcal{O})$ be a ringed ∞-topos, let $\mathfrak{X} = \tau_{\leq 0} \mathcal{X}$ be the underlying topos of discrete objects. Then $(\mathcal{X}, \mathcal{O})$ is a $\mathcal{T}_{\text{ét}}$-derived scheme if and only if the following conditions are satisfied:*

1. *The ringed topos $(\mathfrak{X}, \pi_0 \mathcal{O})$ is a Deligne-Mumford stack.*
2. *The adjunction morphism $\mathcal{X} \to \Delta \mathfrak{X}$ is étale.*
3. *Each of the sheaves $\pi_i \mathcal{O}$ on $\mathfrak{X}$ is quasi-coherent.*
4. *The structure sheaf $\mathcal{O}$ is a hypersheaf.*

Proof. First suppose that $(\mathcal{X}, \mathcal{O})$ is a $\mathcal{T}_{\text{ét}}$-derived scheme. Then (2) holds by Theorem 4.5.10. To prove (1), (3), and (4) it suffices to work locally on $\mathcal{X}$, so that we may assume that $\mathcal{X} = \text{Spec}^{\mathcal{T}_{\text{ét}}} A$ where $A \in \mathcal{SCR}$. Then $(\mathfrak{X}, \pi_0 \mathcal{O})$ is equivalent, as a ringed topos, to the étale spectrum of $\pi_0 A$, which is a Deligne-Mumford stack. Each $\pi_i \mathcal{O}$ is the quasi-coherent sheaf of modules associated to the $\pi_0 A$-module $\pi_i A$. The structure sheaf $\mathcal{O}$ is a hypersheaf because it is the inverse limit of its Postnikov tower $\{\tau_{\leq n} \mathcal{O}\}$ (which is easy to check on the "basic affine" opens in $\mathfrak{X}$).

For the converse, let us suppose that $(\mathcal{X}, \mathcal{O})$ satisfies (1), (2), (3) and (4). We must show that $(\mathcal{X}, \mathcal{O})$ is a $\mathcal{T}_{\text{et}}$-derived scheme. This assertion is local on $\mathcal{X}$, so that using (2) we may reduce to the case where $\mathcal{X} \to \Delta \mathfrak{X}$ is an equivalence. Localizing further, we may suppose that $(\mathfrak{X}, \pi_0 \mathcal{O})$ is affine. Let X denote the final object of $\mathfrak{X}$. Let $A = \mathcal{O}(X) \in \mathcal{SCR}$. Since $(\mathcal{X}, \tau_{\leq 0} \mathcal{O})$ is $\mathcal{T}_{\text{ét}}$-local, we deduce that $(\mathcal{X}, \mathcal{O})$ is $\mathcal{T}_{\text{ét}}$-local so that the universal property of $\text{Spec}\, A$ furnishes a $\mathcal{T}_{\text{ét}}$-local morphism

$$\pi : (\mathcal{X}, \mathcal{O}) \to \text{Spec}\, A.$$

We must show that π is an equivalence.

Let $\mathcal{O}_\infty$ denote the inverse limit of the Postnikov tower

$$\ldots \to \tau_{\leq 1}\,\mathcal{O} \to \tau_{\leq 0}\,\mathcal{O}\,.$$

Then $\mathcal{O}_\infty$ is a hypersheaf, and there is a natural morphism $p : \mathcal{O} \to \mathcal{O}_\infty$. We claim that p is an equivalence. By assumption (4), it suffices to show that p is ∞-connected. In other words, we must show that $\tau_{\leq n}\,\mathcal{O} \to \tau_{\leq n}\,\mathcal{O}_\infty$ is an equivalence for each n. It suffices to check this over each affine; shrinking $\mathcal{X}$ if necessary, we reduce to proving that $\pi_i(\tau_{\leq n}\,\mathcal{O})(X) \simeq \pi_i\,\mathcal{O}_\infty(X)$ for $i \leq n$. Since $\mathcal{O}_\infty(X)$ is given as the limit of the sequence $\{(\tau_{\leq m}\,\mathcal{O})(X)\}$, we see from the appropriate long exact sequence that it suffices to prove that the sequences $\{\pi_n((\tau_{\leq m}\,\mathcal{O})(X))\}$ and $\{\pi_{n+1}((\tau_{\leq m}\,\mathcal{O})(X))\}$ are constant for $m > n$. On the other hand, we may compute $\pi_n((\tau_{\leq m}\,\mathcal{O})(X))$ using a spectral sequence with E_2^{pq}-term $\mathrm{H}^p(\mathcal{X}, \pi_q(\tau_{\leq m}\,\mathcal{O}))$. By assumption (3) and Grothendieck's vanishing theorem, this spectral sequence is degenerate and we get $\pi_i((\tau_{\leq n}\,\mathcal{O})(X)) = (\pi_i\,\mathcal{O})(X)$ for $n \geq i$, which does not depend on n.

Fix $n \geq 0$, and let $A_n = (\tau_{\leq n}\,\mathcal{O})(X) \in \mathcal{SCR}$. Then A_n is n-truncated, so that there is a natural map $\psi_n : \tau_{\leq n}A \to A_n$. Using the degenerate spectral sequence considered above, one shows that ψ_n is an equivalence. In particular, the ring of global sections $(\pi_0\,\mathcal{O})(X)$ is naturally isomorphic to $\pi_0 A$. Since the ∞-topos $\mathcal{X}$ is equivalent to the étale ∞-topos of $(\pi_0\,\mathcal{O})(X) = \pi_0 A$, which is the ∞-category of $\mathcal{S}$-valued sheaves on the étale topos of A, we see that π induces an equivalence on the underlying ∞-topoi. Moreover, the above computations show that the natural map $\pi^*\,\mathcal{O}_{\operatorname{Spec} A} \to \mathcal{O}$ is ∞-connected. Since both sides are hypersheaves, π is an equivalence. □

Remark 4.5.17. Condition (4) of Theorem 4.5.16 could be omitted if we were to work with t-complete ∞-topoi: in that case, *any* sheaf is a hypersheaf.

Remark 4.5.18. It follows from the proof of Theorem 4.5.16 that a derived scheme is $(\mathcal{X}, \mathcal{O})$ is affine if and only if $\mathcal{X} \simeq \Delta\tau_{\leq 0}\,\mathcal{X}$ and the Deligne-Mumford stack $(\tau_{\leq 0}, \pi_0\,\mathcal{O})$ is affine.

Remark 4.5.19. We may interpret Theorem 4.5.16 as showing that our notion of a derived scheme is not excessively general. It is, in some sense, the simplest generalization of ordinary Deligne-Mumford stacks which simultaneously allows for "higher orbifold behavior" and "higher-order infinitesimals" in the structure sheaf.

4.6 Functors Representable by Derived Schemes

The objective of this section is to show that we may view the ∞-category of derived schemes as a full subcategory of the ∞-category of sheaves on the big étale site of $\mathcal{SCR}$. The first step is to show that derived schemes actually form an ∞-category (in other words, that the ∞-category of morphisms $\mathrm{Hom}(X, Y)$ between two derived schemes is actually a small ∞-groupoid). More generally, we have the following:

Proposition 4.6.1. *Let $\mathcal{T}$ denote the étale topology (the Zariski or Nisnevich topologies would work easily well). Let X be a $\mathcal{T}$-local ringed ∞-topos, and let Y be a $\mathcal{T}$-derived scheme. Then $\mathrm{Hom}_{\mathcal{T}}(X,Y)$ is a small ∞-groupoid.*

Proof. First, suppose that $Y = \operatorname{Spec} A$. Then, by definition, $\mathrm{Hom}_{\mathcal{T}}(X,Y) = \mathrm{Hom}_{\mathcal{SCR}}(A, \mathcal{O}(E))$, where $X = (\mathcal{X}, \mathcal{O})$ and $E \in \mathcal{X}$ is the final object.

Now suppose that Y is a disjoint union of affine derived schemes $\{Y_\alpha\}_{\alpha \in I}$. In this case, $\mathrm{Hom}_{\mathcal{T}}(\mathcal{X}, \mathcal{Y})$ is a disjoint union of ∞-categories $\Pi_\alpha \mathrm{Hom}_{\mathcal{T}}(X_\alpha, Y_\alpha)$, where the union is taken over all decompositions of X as a disjoint union of components $\{X_\alpha\}_{\alpha \in I}$. Since there are only a bounded number of such decompositions, we see that $\mathrm{Hom}_{\mathcal{T}}(\mathcal{X}, \mathcal{Y})$ is a small ∞-groupoid.

Now suppose that the theorem is known for $Y = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$, and let $U \in \mathcal{Y}$ be any object. Then $\mathrm{Hom}_{\mathcal{T}}(X, (\mathcal{Y}_{/U}, \mathcal{O}_{\mathcal{Y}}|U)) \to \mathrm{Hom}_{\mathcal{T}}(X,Y)$ is a fibration of ∞-categories with fiber over $f : X \to Y$ given by the space $\mathrm{Hom}_{\mathcal{X}}(E, f^*U)$. Since the base and fiber are both small ∞-groupoids, so is $\mathrm{Hom}_{\mathcal{T}}(X,Y)$.

We now pass to the general case. Since $Y = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is a derived scheme, there exists $U_0 \in \mathcal{Y}$ such that $(\mathcal{Y}_{/U_0}, \mathcal{O}_{\mathcal{Y}}|U_0)$ is a disjoint union of affine $\mathcal{T}$-derived schemes. Let U_n denote the $(n+1)$-fold power of U_0, and let $Y_n = (\mathcal{Y}_{/U_n}, \mathcal{O}_{\mathcal{Y}}|U_n)$. For each $V \in \mathcal{X}$, let X_V denote the ringed ∞-topos $(\mathcal{X}_{/V}, \mathcal{O}|V)$. Then each Y_n is a derived scheme, étale over Y_0, so that $\mathrm{Hom}_{\mathcal{T}}(X_V, Y_n)$ is a small ∞-groupoid for each n, for any $V \in \mathcal{X}$. One shows that the ∞-category valued sheaf

$$V \mapsto \mathrm{Hom}_{\mathcal{T}}(X_V, Y)$$

is the sheafification of the ∞-category valued presheaf

$$V \mapsto |\mathrm{Hom}_{\mathcal{T}}(X_V, Y_\bullet)|.$$

Since each $\mathrm{Hom}_{\mathcal{T}}(X_V, Y_n)$ is a small ∞-groupoid, the same is true of the geometric realization and the desired result follows. □

Consequently, we deduce that the $(\infty,2)$-category of $\mathcal{T}$-derived schemes is actually an ordinary ∞-category (all of its Hom-categories are in fact small ∞-groupoids).

Now let $(\mathcal{X}, \mathcal{O})$ be a derived scheme. Then $(\mathcal{X}, \mathcal{O})$ determines a covariant functor $\mathcal{SCR} \to \mathcal{S}$, given by

$$A \mapsto \mathrm{Hom}_{\mathcal{T}}(\operatorname{Spec} A, (\mathcal{X}, \mathcal{O})).$$

We may regard this correspondence as defining a functor R from the $(\infty,2)$-category of derived schemes to the ∞-category $\mathcal{S}^{\mathcal{SCR}}$.

Proposition 4.6.2. *The functor R is fully faithful.*

Proof. Let X and Y be derived schemes. We must show that $\mathrm{Hom}_{\mathcal{T}}(X,Y) \to \mathrm{Hom}_{\mathcal{S}^{\mathcal{SCR}}}(RX, RY)$ is an equivalence. One first shows that both sides are sheaves on the underlying ∞-topos of X. Thus we can reduce to the case where $X = \operatorname{Spec} A$. In this case, the left hand side is $RY(A)$ by definition. The equivalence $RY(A) \simeq \mathrm{Hom}_{\mathcal{S}^{\mathcal{SCR}}}(R \operatorname{Spec} A, RY)$ follows from the proof of Yoneda's lemma. □

Consequently, it is safe to identify derived schemes with the functors on $\mathcal{SCR}$ which they represent. We now remark that this class of functors has good closure properties:

Proposition 4.6.3. *The ∞-category of derived schemes has all finite limits.*

Proof. It suffices to construct fiber products and a final object. The final object is $\operatorname{Spec} \mathbf{Z}$. To construct fiber products $X \times_Y Z$, it suffices to work locally on X, Y, Z. Then we can reduce to the case $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $Z = \operatorname{Spec} C$, so that $X \times_Y Z = \operatorname{Spec} A \otimes_B C$. $\square$

We end this section by giving a characterization of quasi-coherent complexes on a derived scheme. Recall that if $\mathcal{F} : \mathcal{SCR} \to \mathcal{S}$ is any functor, then we have defined a *quasi-coherent complex* M on $\mathcal{F}$ to be a functor which assigns to each $\eta \in \mathcal{F}(C)$ a C-module $M(\eta)$, which is functorial in η in the strong sense that a map $\gamma : C \to C'$ induces an equivalence

$$M(\gamma_* \eta) \simeq M(\eta) \otimes_C C'.$$

Let us now specialize to the case where $\mathcal{F}(C) = \operatorname{Hom}(\operatorname{Spec} C, X)$, where X is a derived scheme. Our aim is to show that the abstract definition given above is equivalent to a more concrete notion, involving sheaves of modules on the underlying ∞-topos of X.

If $(\mathcal{X}, \mathcal{O})$ is a ringed ∞-topos, then we shall denote by $\mathcal{M}_{\mathcal{O}}$ the ∞-category of sheaves of $\mathcal{O}$-modules on $\mathcal{X}$. In other words, an object M of $\mathcal{M}_{\mathcal{O}}$ assigns to each $E \in \mathcal{X}$ a $\mathcal{O}(E)$-module $M(E)$, such that underlying presheaf of spectra on $\mathcal{X}$ is actually a sheaf of spectra. If $M \in \mathcal{M}_{\mathcal{O}}$, then we let $\pi_i M$ denote the sheafification of the presheaf

$$V \mapsto \pi_i M(V).$$

This is a sheaf of discrete $\pi_0 \mathcal{O}$-modules on $\mathcal{X}$. We shall say that M is a *hypersheaf* if the sheaf of spaces $E \mapsto \Omega^\infty M(E)$ is a hypersheaf, where $\Omega^\infty : \mathcal{S}_\infty \to \mathcal{S}$ denotes the "zeroth space" functor.

Lemma 4.6.4. *Assume that $\mathcal{T}$ is the étale topology. Let $A \in \mathcal{SCR}$, let $\operatorname{Spec} A = (\mathcal{X}, \mathcal{O})$, and define $\phi : \mathcal{M}_A \to \mathcal{M}_{\mathcal{O}}$ by letting $\phi(M)$ denote the sheafification of the presheaf*

$$V \mapsto M \otimes_A \mathcal{O}(V).$$

The functor ϕ is fully faithful. Its essential image consists of those $\mathcal{O}$-modules $\widetilde{M}$ satisfying the following conditions:

- *Each $\pi_i \widetilde{M}$ is a quasi-coherent sheaf on the Deligne-Mumford stack $(\tau_{\leq 0} \mathcal{X}, \pi_0 \mathcal{O})$.*
- *The sheaf of $\mathcal{O}$-modules $\widetilde{M}$ is a hypersheaf.*

Proof. The functor ϕ has a right adjoint, given by $\widetilde{M} \mapsto \widetilde{M}(E)$, where $E \in \mathcal{X}$ is the final object. Consequently, ϕ is exact and commutes with all colimits. To prove that

$$\operatorname{Hom}_{\mathcal{M}_A}(M, N) \to \operatorname{Hom}_{\mathcal{M}_{\mathcal{O}}}(\phi M, \phi N)$$

is an equivalence, we may reduce to the case where $M = A$. Then the left hand side is the zeroth space of N and the right hand side is the zeroth space of $(\phi N)(E)$. By construction, ϕN is the sheafification of a presheaf whose value on $\operatorname{Spec} B$ is given by $B \otimes_A N$ for any étale A-algebra B. To complete the proof, it suffices to note that sheafification does not affect the value of ϕN on any object $U \in \mathcal{X}$ which is the spectrum of an étale A-algebra. In other words, it suffices to show that the functor

$$B \mapsto B \otimes_A N$$

is an étale sheaf on $\mathcal{SCR}^{op}$. In fact, it is a sheaf with respect to the flat hypertopology.

The proof of the characterization of the essential image of ϕ is analogous to the proof of Theorem 4.5.16. □

Theorem 4.6.5. *Let $X = (\mathcal{X}, \mathcal{O})$ be a derived scheme for the étale topology. Then the ∞-category QC_X is equivalent to the full subcategory of $\mathcal{M}_{\mathcal{O}}$ consisting of those $\mathcal{O}$-modules $\widetilde{M}$ satisfying the following conditions:*

- *Each $\pi_i \widetilde{M}$ is a quasi-coherent sheaf on the Deligne-Mumford stack $(\tau_{\leq 0} \mathcal{X}, \pi_0 \mathcal{O})$.*
- *The sheaf of $\mathcal{O}$-modules $\widetilde{M}$ is a hypersheaf.*

Proof. One shows that the statement is local on $\mathcal{X}$ and therefore reduces to Lemma 4.6.4. □

Remark 4.6.6. As with Theorem 4.5.16, we can dispense with the hypersheaf condition if we choose to work with t-complete ∞-topoi.

We will henceforth identify quasi-coherent complexes on derived schemes $(\mathcal{X}, \mathcal{O})$ with the corresponding sheaves of $\mathcal{O}$-modules.

Warning 4.6.7. If $(\mathcal{X}, \mathcal{O})$ is a derived scheme, then the assignment $U \mapsto \mathcal{O}(U)$ is *not* necessarily a quasi-coherent complex on $\mathcal{X}$. The reason is that limits in $\mathcal{SCR}$ are not necessarily compatible with limits of the underlying spectra. When we wish to view the structure sheaf as a quasi-coherent complex, we must first sheafify it. The resulting sheaf M of $\mathcal{O}$-modules agrees with $\mathcal{O}$ on any affine U, essentially because of Grothendieck's theorem on the vanishing of the cohomology of quasi-coherent sheaves on affine schemes. On a general $U \in \mathcal{X}$, the group $\pi_{-i}(M(U)) = \mathrm{H}^i(U, \mathcal{O})$ is a hypercohomology group of the structure sheaf. We may occasionally abuse notation by referring to the structure sheaf $\mathcal{O}$ as a quasi-coherent complex; in this case we are really referring to the sheafification of $\mathcal{O}$ as an $\mathcal{O}$-module.

Chapter 5

Derived Stacks

The purpose of this section is to develop the "derived analogue" of the theory of Artin stacks. We recall that an Artin stack is defined to be a groupoid-valued functor on the category of commutative rings which is a sheaf for the étale topology and which is in some sense "locally" representable by a scheme with respect to the smooth topology. In the derived context the definition is similar, except that we replace the ordinary category of commutative rings by the ∞-category $\mathcal{SCR}$. In this setting the notion of a "groupoid-valued functor" is too restrictive: even for an affine derived scheme X, the space $\mathrm{Hom}(\mathrm{Spec}\, A, X)$ may have homotopy groups in arbitrarily high dimensions. Consequently, we must deal with $\mathcal{S}$-valued functors everywhere. Granting this, it is natural for our theory to encompass also "higher Artin stacks", which represent higher-groupoid-valued functors even on ordinary commutative rings. These higher Artin stacks arise naturally in a number of situations. For example, one may consider the "n-fold classifying stack of the additive group" $Y = K(\mathbf{G}_a, n)$, which has the property that $\pi_0 \mathrm{Hom}(X, Y) \simeq \mathrm{H}^n(X, \mathcal{O}_X)$.

We begin in §5.1 with a definition of derived stacks. The next section, §5.2, contains a quick discussion of quasi-coherent complexes on derived stacks. In §5.3 we introduce a few of the more important conditions which may be imposed on derived stacks and their morphisms.

In order to compare derived stacks with their classical analogues, we shall develop in §5.4 a mechanism for analyzing an arbitrary derived stack X as the direct limit of its "n-truncations" $\tau_{\leq n} X$ (which, for $n = 0$, is determined by functor that X represents on ordinary commutative rings).

In §5.5, we prove two different analogues of the Grothendieck-Serre theorem on the coherence of proper higher direct image of coherent sheaves.

Finally, in §5.6, we study the operation of "gluing" two derived schemes together along a common closed subscheme.

5.1 Definition of Derived Stacks

In the last section, we constructed a derived analogue of the ordinary theory of schemes (and, more generally, Deligne-Mumford stacks). We now enlarge the scope of our investigation to include a larger class of functors, which we call we call *derived Artin stacks*. In order to make this generalization, we will abandon the idea that our geometric objects should be given by some kind of space with a sheaf of rings, and instead consider them to be $\mathcal{S}$-valued functors on $\mathcal{SCR}$ (which is a viable approach in the case of derived schemes by the results of §4.6).

We will denote by $\mathrm{Shv}(\mathcal{SCR}^{op})$ the ∞-category of covariant functors $\mathcal{SCR} \to \mathcal{S}$ which are sheaves with respect to the étale topology. We shall think of these as being represented by moduli spaces. We shall say that a map $X \to Y$ of objects of $\mathrm{Shv}(\mathcal{SCR}^{op})$ is *surjective* if it is a surjection of étale sheaves: that is, for any $A \in \mathcal{SCR}$ and any $\eta \in Y(A)$, there exists an étale covering $\{A \to A_\alpha\}$ and liftings $\widetilde{\eta}_\alpha \in X(A_\alpha)$ of $\eta | A_\alpha$.

We would like to say that $X \in \mathrm{Shv}(\mathcal{SCR}^{op})$ is a derived stack if, in some sense, it is locally represented by an affine derived scheme with respect to the smooth topology on $\mathcal{SCR}$. More precisely, we should assume the existence of a "smooth" surjection $p : U \to X$, where U is a disjoint union of affine derived schemes. Of course, in order to make sense of the smoothness of p, we need to assume that the fibers of the morphism p form some reasonable sort of geometric object; in other words, that they are already derived stacks. Consequently, our definition will have an inductive character. We should begin with some subcategory $S_0 \subseteq \mathrm{Shv}(\mathcal{SCR}^{op})$, which we call *0-stacks*, and then inductively define S_{n+1} to be the class of functors X admitting a smooth surjection $p : U \to X$, where U is a disjoint union of affine derived schemes and every fiber $U \times_X \operatorname{Spec} A$ of p lies in S_n. We are therefore faced with a question of where to begin the induction: that is, what is the right class S_0 of 0-stacks? There are at least three reasonable candidates:

- The most conservative choice would be let S_0 be the class of corepresentable functors on $\mathcal{SCR}$: in other words, the class of affine derived schemes. In this case, S_n would be analogous to the class of *n-geometric* stacks in the sense of [36]. This is an important notion which we will return to in [24].

- To conform with the standard terminology in algebraic geometry, we could take S_0 to be some derived analogue of the class of algebraic spaces. More specifically, we could let S_0 denote the class of affine derived schemes $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ for which $\mathcal{X} \simeq \Delta \tau_{\leq 0} \mathcal{X}$ and the pair $(\tau_{\leq 0} \mathcal{X}, \tau_{\leq 0} \mathcal{O}_\mathcal{X})$ is an algebraic space. In this case, the defining property of S_n is that an object of S_n takes n-truncated values when restricted to ordinary commutative rings.

- We could take S_0 to be the class of *all* functors which are representable by derived schemes. In this case, a derived stack belongs to S_n if and only if its cotangent complex is $(-n-1)$-connected (see Theorem 5.1.12). This has the advantage of leading to a larger class of objects than the previous alternatives, which includes all derived schemes. However, this extra generality does not seem to be of much practical use.

We shall follow the second course in our definition of a derived stack. Either of the others is possible, although if the third is adopted then some of the results that we shall prove are only valid under certain restrictions.

Remark 5.1.1. For us, the notion of an algebraic space is more general than the definition given in [18], since we do not require local quasi-separatedness, or that the diagonal of an algebraic space is a scheme. However, if $X \to Y$ is a relative derived algebraic space in our sense, then the diagonal $X \to X \times_Y X$ is always a relative derived algebraic space in the stronger sense, since its underlying (ordinary) algebraic space is actually *separated.*

Proposition 5.1.2. *Let $X = (\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a derived scheme. The following conditions are equivalent:*

1. *The natural map of ∞-topoi $\mathcal{X} \to \Delta\tau_{\leq 0}\mathcal{X}$ is an equivalence, and the Deligne-Mumford stack $(\tau_{\leq 0}\mathcal{X}, \tau_{\leq 0}\mathcal{O}_\mathcal{X})$ is an algebraic space.*

2. *The space* $\mathrm{Hom}(\operatorname{Spec} A, X)$ *is discrete whenever A is an ordinary commutative ring.*

Proof. If (1) is satisfied, then X and $(\tau_{\leq 0}\mathcal{X}, \tau_{\leq 0}\mathcal{O}_\mathcal{X})$ represent the same functor on commutative rings, which is set-valued since the latter is an algebraic space.

Conversely, let us suppose assume that (2) is valid. Let Y be the Deligne-Mumford stack $(\tau_{\leq 0}\mathcal{X}, \tau_{\leq 0}\mathcal{O}_\mathcal{X})$. We may write $\mathcal{X} \simeq (\Delta\tau_{\leq 0}\mathcal{X})_{/E}$ for some 1-connected object $E \in \Delta\tau_{\leq 0}\mathcal{X}$. It then follows that for A discrete, we have a natural map $X(A) \to Y(A)$ whose fiber over a map $f : \operatorname{Spec} A \to Y$ is given by the space of global sections of f^*E. Since $X(A)$ is discrete and $Y(A)$ is 1-truncated, the fiber is itself discrete. Since this is also true for any étale A-algebra, we deduce that f^*E is discrete. Since f^*E is 1-connected, it follows that E is final so that $\mathcal{X} \simeq \Delta\tau_{\leq 0}\mathcal{X}$. Thus, when A is discrete, $Y(A) \simeq X(A)$ is discrete. It follows by definition that the Deligne-Mumford stack Y is an algebraic space. □

We shall say that a derived scheme X is a *derived algebraic space* if it satisfies the equivalent conditions of Proposition 5.1.2.

Definition 5.1.3. A morphism $p : X \to Y$ in $\mathrm{Shv}(\mathcal{SCR}^{op})$ is a *relative* 0*-stack* if, for any map $\operatorname{Spec} A \to Y$, the fiber product $\operatorname{Spec} A \times_Y X$ is a derived algebraic space. We shall say that p is *smooth* if each of the associated maps $\operatorname{Spec} A \times_Y X \to \operatorname{Spec} A$ is smooth as a morphism of derived schemes.

For $n > 0$, a morphism $p : X \to Y$ in $\mathrm{Shv}(\mathcal{SCR}^{op})$ is a *relative n-stack* if for any map $\operatorname{Spec} A \to Y$ there exists a smooth surjection $U \to \operatorname{Spec} A \times_Y X$ which is a relative $(n-1)$-stack, where U is a disjoint union of affine derived schemes. We say that p is *smooth* if U may be chosen smooth over Z.

The following bit of temporary terminology will prove useful in proving basic stability properties of the notion of derived stacks: let us say that a morphism $U \to X$ is an *n-submersion* if it is a relative n-stack which is smooth and surjective.

We next verify the basic properties of relative stacks:

Proposition 5.1.4. *1. Any relative n-stack is also a relative m-stack for any $m \geq n$.*

2. Any equivalence is a relative 0-stack.

3. Any morphism homotopic to a morphism which is a relative n-stack is itself a relative n-stack.

4. Any pullback of a (smooth) relative n-stack is a (smooth) relative n-stack.

5. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a composable pair of morphisms. If both f and g are (smooth) relative n-stacks, then so is $g \circ f$.

6. Suppose that $n > 0$ and that we are given morphisms $X \to Y \to Z$, where $X \to Y$ is an $(n-1)$-submersion and $X \to Z$ is a relative n-stack. Then $Y \to Z$ is a relative n-stack.

7. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a composable pair of morphisms and $n \geq 1$. If $g \circ f$ is a relative $(n-1)$-stack and g is a relative n-stack, then f is a relative $(n-1)$-stack.

Proof. Claims (1) through (4) are obvious.

We now prove (5). The proof goes by induction on n. Suppose that $X \to Y$ and $Y \to Z$ are both relative n-stacks. We may assume without loss of generality that Z is affine. If $n = 0$, then the hypotheses imply that Y is a derived algebraic space. Working locally on Y, we deduce that X is a derived scheme. For any ordinary commutative ring A, $Y(A)$ is discrete and the fibers of the map $X(A) \to Y(A)$ are discrete, so that $X(A)$ is discrete. It follows that X is a derived algebraic space, as required.

If $n > 0$, then we may choose an $(n-1)$-submersion $U \to Y$, where U is a disjoint union of affine derived schemes. Then the base change $U \times_Y X \to X$ is an $(n-1)$-submersion. Similarly, there is an $(n-1)$-submersion $V \to U \times_Y X$, where V is a disjoint union of affine derived schemes. The inductive hypothesis implies that $f : V \to X$ is an $(n-1)$-submersion, and the conclusion follows. If f and g are both smooth, then we may choose U smooth over Z and V smooth over U, hence V smooth over Z.

We next prove (6). Without loss of generality we may suppose that Z is affine. By hypothesis, there exists an $(n-1)$-submersion $U \to X$, where U is a disjoint union of affine derived schemes. Then the composite map $U \to Y$ is an $(n-1)$-submersion by (5), so that $Y \to Z$ is a relative n-stack.

It remains to prove (7). Once again, we may suppose that Z is affine. Suppose first that $n = 1$. Then X is a derived algebraic space. Let $\operatorname{Spec} A \to Y$ be any morphism; we must show that $X \times_Y \operatorname{Spec} A$ is a derived algebraic space. It is clear that $X \times_Y \operatorname{Spec} A$ takes discrete values on discrete commutative rings, so it suffices to show that $X \times_Y \operatorname{Spec} A$ is a derived scheme. This assertion is local on X and $\operatorname{Spec} A$, so we may assume the existence of factorizations $X \to U$, $\operatorname{Spec} A \to U$, where $U \to Y$ is a 0-submersion and U is a derived algebraic space. In this case, $X \times_Y \operatorname{Spec} A = X \times_U (U \times_Y U) \times_U \operatorname{Spec} A$, so it suffices to show that $U \times_Y U$ is a relative algebraic space. This follows immediately from the definition of a 0-submersion.

Now suppose that $n > 1$. Choose an $(n-2)$-submersion $U \to X$ where U is a disjoint union of affine derived schemes. Using part (6), it will suffice to show that $p : U \to Y$ is a relative $(n-1)$-stack. Let $q : V \to Y$ be an $(n-1)$-submersion, where V is a derived algebraic space. The assertion that p is a relative $(n-1)$-stack is local on U; since q is surjective we may suppose that there exists a factorization $U \to V \to Y$, and it suffices to show that the map $U \to V$ is a relative 0-stack. This follows immediately from the fact that both U and V are derived algebraic spaces. □

We next study the cotangent complexes of derived stacks.

Proposition 5.1.5. *Let $f : X \to Y$ be a relative n-stack. Then f has a cotangent complex $L_{X/Y}$. Moreover, $L_{X/Y}[n]$ is connective. If f is smooth, then $L_{X/Y}$ is the dual of a connective, perfect complex.*

Proof. The proof goes by induction on n. Without loss of generality, $Y = \operatorname{Spec} A$ is affine. If $n = 0$, then X is a derived algebraic space. The assertion that $L_{X/Y}$ exists is local on X, so we may suppose that X is affine; in this case it follows from Proposition 3.2.14.

Now suppose $n > 0$. Choose an $(n-1)$-submersion $U \to X$, where U is a disjoint union of affine derived schemes. We must show that for any $\eta \in X(B)$, the functor which carries $M \in \mathcal{M}_B$ to the mapping fiber of

$$X(B \oplus M) \to X(B) \times_{Y(B)} Y(B \oplus M)$$

is corepresentable by some $(-1-n)$-connected module $L_{X/Y}(\eta) \in \mathcal{M}_B$, and that $L_{X/Y}(\eta)$ is compatible with base change. Let us denote the mapping fiber in question by $\Omega(X, Y, \eta, M)$.

Both assertions are local with respect to the topology $\mathcal{T}$, so we may suppose that η admits a lifting $\widetilde{\eta} \in U(B)$. Shrinking U, we may suppose that U is affine (possibly U no longer surjects onto X, but we no longer need this). Then we have a natural map of spaces

$$p : \Omega(U, Y, \widetilde{\eta}, M) \to \Omega(X, Y, \eta, M).$$

The mapping fiber of p over $0 \in \Omega(X, Y, \eta, M)$ is naturally equivalent to $\Omega(U, X, \eta, M)$. By the inductive hypothesis, $\Omega(U, X, \widetilde{\eta}, M)$ is corepresented (as a functor of M) by $L_{U/X}(\widetilde{\eta})$. Since U and Y are affine, $\Omega(U, Y, \widetilde{\eta}, M)$ is corepresentable by $L_{U/Y}(\widetilde{\eta})$. We note that the fiber sequence

$$\Omega(U, X, \widetilde{\eta}, M) \to \Omega(U, Y, \widetilde{\eta}, M) \to \Omega(X, Y, \eta, M)$$

deloops. Since p is surjective (because U is formally smooth over X), we may identify $\Omega(X, Y, \eta, M)$ with the mapping fiber of the map of connected deloopings $\Omega^{-1} \operatorname{Hom}(L_{U/X}(\widetilde{\eta}), M) \to \Omega^{-1} \operatorname{Hom}(L_{U,Y}(\widetilde{\eta}, M))$. Since $L_{U/X}(\widetilde{\eta})$ is the dual of a connective, perfect complex, the first connected delooping is given by $\operatorname{Hom}(L_{U/X}(\widetilde{\eta}, M[1]))$, so that $\Omega(X, Y, \eta, M)$ may also be identified with the mapping fiber of

$$\operatorname{Hom}(L_{U/X}(\widetilde{\eta}), M[1]) \to \operatorname{Hom}(L_{U/Y}(\widetilde{\eta}, M[1])).$$

By definition, this mapping fiber is corepresentable by the cokernel of $L_{U/Y}(\widetilde{\eta})[-1] \to L_{U/X}(\widetilde{\eta})[-1]$, which is also the kernel of $L_{U/Y}(\widetilde{\eta}) \to L_{U/X}(\widetilde{\eta})$. This proves the existence of $L_{X/Y}(\eta)$ and the compatibility with base change (since $L_{U/Y}$ and $L_{U/X}$ are both compatible with base change). It also shows that $L_{X/Y}[n]$ is connective (since $L_{U/Y}$ is connective and $L_{U/X}[n-1]$ is connective by the inductive hypothesis).

If $X \to Y$ is smooth, then we may suppose that $L_{U/Y}(\widetilde{\eta})$ is projective and finitely generated, and therefore has a connective predual. Then $L_{X/Y}(\eta)$ has a connective predual, given by the cokernel of $L^{\vee}_{U/Y}(\widetilde{\eta}) \to L^{\vee}_{U/X}(\widetilde{\eta})$. □

It follows from Proposition 5.1.5 that any smooth relative stack is actually formally smooth.

So far, we have discussed a relative notion of n-stack, since this was better suited to the inductive nature of the definition. Now that we have sorted out the basic facts, it is time to introduce the *absolute* version of this notion. We will say that $X \in \mathrm{Shv}(\mathcal{SCR}^{op})$ is a *derived n-stack* if $X \to \mathrm{Spec}\,\mathbf{Z}$ is a relative n-stack.

Remark 5.1.6. An object $X \in \mathrm{Shv}(\mathcal{SCR}^{op})$ is a derived 0-stack if and only if it is a derived algebraic space.

Remark 5.1.7. A morphism $X \to \mathrm{Spec}\,A$ is a relative n-stack if and only if X is a n-stack. More generally, a morphism $X \to Y$ is a relative n-stack if and only if every fiber product $X \times_Y \mathrm{Spec}\,A$ is a derived n-stack.

Proposition 5.1.8. *Let X be a derived n-stack. Then $X(A)$ is n-truncated for every discrete commutative ring A.*

Proof. The proof of the first assertion goes by induction on n. For $n = 0$, the result is immediate from the definition. Suppose that $n > 0$, and let $x \in X(A)$ be a point. Choose an $(n-1)$-submersion $p : U \to X$, where U is a derived algebraic space. The assertion in question is local on A, so we may suppose that x admits a lift $\widetilde{x} \in U(A)$. The space $U(A)$ is discrete, and the inductive hypothesis implies that the fibers of the map $U(A) \to X(A)$ are $(n-1)$-truncated. It follows that the connected component of x in $X(A)$ is n-truncated. □

We also have the following:

Proposition 5.1.9. *Let $X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a derived scheme and $n \geq 1$. The following conditions are equivalent:*

1. *As an object of $\mathrm{Shv}(\mathcal{SCR}^{op})$, X is an n-stack.*

2. *If A is a discrete commutative ring, then $\mathrm{Hom}(\mathrm{Spec}\,A, X)$ is n-truncated.*

3. *If we write $\mathcal{X} = (\Delta\tau_{\leq 0}\,\mathcal{X})_{/E}$, then E is n-truncated.*

Proof. The implication (1) $\to$ (2) follows from Proposition 5.1.8. The reverse implication is proven by induction on n. Since X is a derived scheme, we may choose an étale surjection $p : U \to X$, where U is a disjoint union of affine derived schemes. Applying the inductive hypothesis to the fibers of p, we deduce that p is an $(n-1)$-submersion so that X is an n-stack.

The proof of the equivalence of (2) and (3) is analogous to the proof of Proposition 5.1.2. □

We shall say that an object $X \in \mathrm{Shv}(\mathcal{SCR}^{op})$ is a *derived stack* if it is a derived n-stack for $n \gg 0$.

Remark 5.1.10. According to the definition above, not every derived scheme X is a derived stack. If $X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, and $\mathcal{X} \simeq (\Delta \tau_{\leq 0} \mathcal{X})_{/E}$, then X is a derived stack if and only if E is truncated. As remarked earlier, we could remove this restriction by starting our inductive definition of stacks by allowing *all* derived schemes. This leads to a little more generality, but also to additional technicalities owing to the poor formal behavior of non-truncated sheaves of spaces in the étale topology.

We have already observed that if X is a derived scheme, then the cotangent complex of X is connective. Our next goal is to prove the converse of this assertion. First, we need a lemma.

Lemma 5.1.11. *Let $p : U \to X$ be a morphism in* $\mathrm{Shv}(\mathcal{SCR}^{op})$, *where U is a derived scheme. Suppose that p is surjective, and that for each morphism* $\mathrm{Spec}\, A \to X$, *the fiber product* $U \times_X \mathrm{Spec}\, A$ *is a derived scheme which is étale over* $\mathrm{Spec}\, A$. *Then X is a derived scheme.*

Proof. Let us begin with any functor $Y \in \mathrm{Shv}(\mathcal{SCR}^{op})$. We shall call a morphism $p : V \to Y$ *étale* if any fiber product $V \times_Y \mathrm{Spec}\, A$ is a derived scheme, étale over $\mathrm{Spec}\, A$.

We will attempt to represent Y by a derived scheme $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$. We define the ∞-topos $\mathcal{Y}$ as follows: the objects of $\mathcal{X}$ are étale morphisms $V \to Y$, where V is a derived scheme. Since any morphism between derived schemes étale over Y is itself étale, one can show with a bit of effort that $\mathcal{Y}$ is an ∞-topos provided that it is accessible. We shall gloss over this technical point (which can be addressed whenever the functor Y is reasonably continuous: in particular, continuity follows from the existence of a surjection $U \to Y$ where U is representable by a derived scheme).

On the ∞-topos $\mathcal{Y}$ there is a tautological $\mathcal{SCR}$-valued sheaf. Namely, we assign to each $(\mathcal{V}, \mathcal{O}_{\mathcal{V}}) \to Y$ the global sections of $\mathcal{O}_{\mathcal{V}}$. It is immediate from the definition that $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is a derived scheme. We shall denote this derived scheme by Y'; it comes equipped with a (-1)-truncated transformation $Y' \to Y$. If Y is a derived scheme, then the natural map gives an equivalence $Y' \simeq Y$. In the general case, Y' is the largest subfunctor of Y which is representable by a derived scheme.

Returning to the situation of the proposition, we wish to show that $X' \simeq X$. It suffices to show that $X' \to X$ is surjective. Since p is surjective, it suffices to show that $U \to X$ factors through U. This is immediate from the definition. □

Theorem 5.1.12. *Let $f : X \to Y$ be a relative stack, and suppose that $L_{X/Y}$ is connective. Then f is a relative derived scheme.*

Proof. The proof is essentially identical to that of the more classical fact that an Artin stack with unramified diagonal is a Deligne-Mumford stack (which is a special case of Theorem 5.1.12).

Without loss of generality, we may suppose that Y is affine. Let X be a relative n-stack; we prove the result by induction on n. If $n = 0$ there is nothing to prove. Otherwise, we may choose an $(n-1)$-submersion $U \to X$, where U is a disjoint union of affine derived schemes. Then we have an exact triangle

$$L_{X/Y}|U \to L_{U/Y} \to L_{U/X},$$

which shows that $L_{U/X}$ is connective. By the inductive hypothesis, $U \to X$ is a relative derived scheme.

We next construct derived scheme U' over U, which we will obtain by "slicing" U. Consider all instances of the following data: étale morphisms $\operatorname{Spec} A \to U$ together with m-tuples $\{a_1, \ldots, a_m\} \subseteq \pi_0 A$ such that $\{da_1, \ldots, da_m\}$ freely generate $\pi_0(L_{U/X}| \operatorname{Spec} A)$. For each such tuple, let A' denote the A-algebra obtained by killing (lifts of) $\{a_1, \ldots, a_m\}$, and let U' denote the derived scheme which is the disjoint union of $\operatorname{Spec} A'$, taken over all A' which are obtained in this way. Then, by construction, $\pi : U' \to X$ is relatively representable by étale morphisms of derived schemes. Lemma 5.1.11 implies that X is a derived scheme, provided that we can show that π is a surjection of étale sheaves. In other words, we must show that for any morphism $\operatorname{Spec} k \to X$ (where $k \in \mathcal{SCR}$), the induced map $\operatorname{Spec} k \times_X U' \to \operatorname{Spec} k$ is a surjection of étale sheaves. We note that $\operatorname{Spec} k \times_X U'$ is a derived scheme which is locally of finite presentation over $\operatorname{Spec} k$ (since $U' \to U$ is locally of finite presentation and $\operatorname{Spec} k \times_X U \to \operatorname{Spec} k$ is smooth). The vanishing of the relative cotangent complex implies that $\operatorname{Spec} k \times_X U'$ is étale over $\operatorname{Spec} k$, so it suffices to prove that the map is surjective on ordinary points. In other words, we may reduce to the case where (as suggested by our notation) k is a field.

Without loss of generality, we may enlarge k and thereby suppose that k is separably closed. Consequently, the map $\operatorname{Spec} k \to X$ factors through a map $f : \operatorname{Spec} k \to \operatorname{Spec} A$, where $\operatorname{Spec} A$ is étale over U. Shrinking $\operatorname{Spec} A$ if necessary, we may suppose that $\pi_0 L_{U/X}(A)$ is freely generated by the differentials of elements $\{x_1, x_2, \ldots, x_m\} \subseteq \pi_0 A$. These elements give rise to an étale morphism from $\operatorname{Spec} A$ to m-dimensional affine space over X which we shall denote by $\mathbf{A}^m_X$. Base changing by the map $\operatorname{Spec} k \to X$, we get an étale morphism of derived schemes $\operatorname{Spec} A \times_X \operatorname{Spec} k \to \mathbf{A}^m_k$. By construction, the former space is nonempty, so the image of this map is some Zariski-open subset of $\mathbf{A}^m_k$. This image therefore contains point with coordinates in k_0, where $k_0 \subseteq k$ denotes the separable closure of the prime field of k. Consequently, we may alter the choice of f and thereby assume that $\{f^*x_1, \ldots, f^*x_m\} \subseteq k_0 \subseteq k$.

Let A' denote the (Zariski) localization of A at the image point of f, and let $\mathfrak{m} \subseteq \pi_0 A'$ denote the maximal ideal. To prove that $\operatorname{Spec} k \to X$ factors through U', it will suffice to show that we choose $\{a_1, \ldots, a_m\} \subseteq \mathfrak{m}$ such that the differentials $\{da_1, \ldots, da_m\}$ freely generate $\pi_0 L_{U/X}(A')$. Since $\pi_0 L_{U/X}(A')$ is free over $\pi_0 A'$, Nakayama's lemma implies that this is equivalent to the surjectivity of the natural map $\pi_1 L_{k/U} = \mathfrak{m}/\mathfrak{m}^2 \to \pi_0 L_{U/X}(k) = \pi_0 L_{\mathbf{A}^m_X/X}(k)$. Using the long exact sequence, we see that this is equivalent to the assertion that the natural map $\pi_0 L_{\mathbf{A}^m_X/X}(k) \to \pi_0 L_{k/X}$ is zero. Since L_X is connective, $\pi_0 L_{k/X}$ is a quotient of $\pi_0 L_k = \Omega_{k/k_0}$. It therefore suffices to show that the differentials of each of the coordinate functions $\{x_1, \ldots, x_m\}$ vanish in Ω_{k/k_0}. This is clear, since the coordinates take their values in k_0 by construction. □

5.2 Quasi-Coherent Complexes on Derived Stacks

Throughout this section, we shall write $\mathcal{T}$ to denote the étale topology on $\mathcal{SCR}$.

Let $X \in \mathrm{Shv}(\mathcal{SCR}^{op})$ be a moduli functor. We have already defined the ∞-category QC_X of quasi-coherent complexes on X. The purpose of this section is to investigate the notion of a quasi-coherent complex in the special case where X is a derived stack.

Let $X \in \mathrm{Shv}(\mathcal{SCR}^{op})$ be a derived stack. Let $Sm_{/X}$ be the ∞-category whose objects are given by pairs (A, ϕ), where $A \in \mathcal{SCR}$ and $\phi : \operatorname{Spec} A \to X$ is a smooth relative stack. Define $\mathcal{M}_X$ to be the strict inverse limit of the ∞-categories $\mathcal{M}_A$, taken over all $(A, \phi) \in Sm_X$. In other words, an object of $\mathcal{M}_X$ assigns functorially to each $A \in Sm_X$ an A-module M_A, and to each factorization $\operatorname{Spec} A \to \operatorname{Spec} B \to X$ an equivalence $M_A \simeq M_B \otimes_B A$.

Using the fact that any map $\operatorname{Spec} B \to X$ locally factors through some smooth relative stack $\operatorname{Spec} A \to X$, one can prove the following:

Lemma 5.2.1. *The restriction functor* $\mathrm{QC}_X \to \mathcal{M}_X$ *is an equivalence of* ∞*-categories.*

Recall that if P is any property of modules which is stable under base change, then P makes sense for quasi-coherent complexes on any X: one asserts that $M \in \mathrm{QC}_X$ has the property P if the A-module $M(\eta)$ has the property P for any $\eta \in X(A)$. In the case where X is a derived stack, we can make sense of this notion more generally:

Definition 5.2.2. Suppose that P is a property of modules which is stable under smooth base change and smooth descent. Let $X \in \mathrm{Shv}(\mathcal{SCR}^{op})$ be a derived stack. We will say that $M \in \mathrm{QC}_X$ has the property P if $p^* M$ has the property P when regarded as an A-module, for any smooth relative stack $\operatorname{Spec} A \to X$.

Proposition 5.2.3. *Let P be a property which is stable under smooth base change and smooth descent, let $p : X \to Y$ be a smooth map between derived stacks, and let $M \in \mathrm{QC}_Y$. If M has the property P, then so does $p^* M$. The converse holds provided that p is surjective.*

Proof. The first claim is clear, since any formally smooth relative stack $\operatorname{Spec} A \to X$ is also a formally smooth relative stack over Y. For the converse, let us suppose that $p^* M$ has the property P, and let $\operatorname{Spec} A \to Y$ be a formally smooth relative stack. Then $\operatorname{Spec} A \times_Y X$ is a relative stack over Z, so there exists a surjective, formally smooth relative stack $U \to \operatorname{Spec} A \times_Y X$ where U is a derived scheme. Without loss of generality, we may assume that

U is a disjoint union of affine pieces. Since p is surjective, the induced map $U \to \operatorname{Spec} A$ is surjective. Replacing U by a union of finitely many open affines, we may suppose that $U = \operatorname{Spec} B$ where B is a faithfully flat, smooth A-algebra. Then $M|\operatorname{Spec} B$ has the property P as a B-module, since $\operatorname{Spec} B \to Y$ factors through a formally smooth morphism to X. Since P satisfies smooth descent, we deduce that $M|\operatorname{Spec} A$ has the property P as an A-module. □

Applying Definition 5.2.2 to the properties of being $(n-1)$-connected and being n-truncated, we define full subcategories $(\mathrm{QC}_X)_{\geq n}$ and $(\mathrm{QC}_X)_{\leq n}$.

Proposition 5.2.4. *Suppose that X is a relative stack over a derived scheme. Then the full subcategories $(\mathrm{QC}_X)_{\geq 0}$ and $(\mathrm{QC}_X)_{\leq 0}$ determine a t-structure on QC_X.*

Proof. It is clear that $(\mathrm{QC}_X)_{\geq 0}$ and $(\mathrm{QC}_X)_{\leq 0}$ are stable under the appropriate shifts. If $M \in (\mathrm{QC}_X)_{\geq 0}$ and $N \in (\mathrm{QC}_X)_{\leq -1}$, then $\operatorname{Hom}_{\mathcal{M}_A}(p^*M, p^*N) = 0$ whenever $p : \operatorname{Spec} A \to X$ is a formally smooth relative stack, so that $\operatorname{Hom}_{\mathrm{QC}_X}(M, N) = 0$ by Lemma 5.2.1. The hard part is to verify that if $M \in \mathrm{QC}_X$, then there exists a triangle

$$M' \to M \to M''$$

with $M' \in (\mathrm{QC}_X)_{\geq 0}$ and $M'' \in (\mathrm{QC}_X)_{\leq -1}$. For this, we apply Lemma 5.2.1 again. For each formally smooth relative stack $p : \operatorname{Spec} A \to X$, we can construct a corresponding triangle of A-modules:

$$M'(p) \to p^*M \to M''(p).$$

To complete the proof, it suffices to show that this triangle is functorial in p. In other words, given any $q : \operatorname{Spec} B \to \operatorname{Spec} A$ such that $q \circ p$ is also a formally smooth relative stack, we must show that the natural map $M'(p) \otimes_A B \to M'(q \circ p)$ is an equivalence.

The fiber product $\operatorname{Spec} A \times_X \operatorname{Spec} B$ is a relative stack which is smooth over both $\operatorname{Spec} A$ and $\operatorname{Spec} B$; since it has a section over $\operatorname{Spec} B$, it surjects onto $\operatorname{Spec} B$. Choose a relative stack and smooth surjection $U \to \operatorname{Spec} A \times_X \operatorname{Spec} B$, where U is a disjoint union of affine derived schemes. Replacing U by a disjoint union of finitely many of these affine derived schemes, we may assume that $U = \operatorname{Spec} C$ and still guarantee that $U \to \operatorname{Spec} B$ is surjective. Then C is an algebra which is smooth over both A and B, and faithfully flat over B. Consequently, it will suffice to show that $M'(p) \otimes_A B \otimes_B C \to M'(q \circ p) \otimes_B C$ is an equivalence. To prove this, one shows that both are equivalent to $M'(r)$, where $r : \operatorname{Spec} C \to X$ is the natural map. This follows from the fact that C is flat over A and B, since tensoring with C carries both of the sequences

$$M'(p) \to p^*M \to M''(p)$$

$$M'(q \circ p) \to (p^*M) \otimes_A B \to M''(q \circ p)$$

into

$$M'(r) \to (p^*M) \otimes_A C \to M''(r).$$

□

5.3 Properties of Derived Stacks

The purpose of this section is to discuss several properties of derived schemes and derived stacks and their interrelationships. The discussion is by no means exhaustive; virtually every notion from classical algebraic geometry has at least one derived analogue.

Throughout this section, $\mathcal{T}$ shall denote the étale topology on $\mathcal{SCR}$.

Definition 5.3.1. Let $A \in \mathcal{SCR}$, $X \in \mathrm{Shv}(\mathcal{SCR}^{op})$, and $p : X \to \operatorname{Spec} A$ a relative 0-stack. We shall say that p is:

- *affine* if $X \simeq \operatorname{Spec} B$ for some A-algebra B.
- a *closed immersion* if $X \simeq \operatorname{Spec} B$ where $\pi_0 A \to \pi_0 B$ is surjective.

If $p : X \to Y$ is a relative derived scheme over an arbitrary base, then we say that p is *affine* (a *closed immersion*) if the induced map $X \times_Y \operatorname{Spec} A \to \operatorname{Spec} A$ has the same property, for any $\operatorname{Spec} A \to Y$.

If P is any property which is stable under base change and local on the source for the étale topology, then we say that a relative derived scheme $p : X \to Y$ has the property P if for any map $\operatorname{Spec} A \to Y$ and any étale morphism $\operatorname{Spec} B \to X \times_Y \operatorname{Spec} A$, the A-algebra B has the property P. Consequently we may speak of morphisms being locally of finite presentation, almost of finite presentation, flat, faithfully flat, étale, smooth, and so forth. If p is a relative derived stack, then we make the same definition for any property which is local for the smooth topology.

We next introduce a compactness condition for relative stacks.

Definition 5.3.2. If $p : X \to Y$ is a relative algebraic space, then we say that p is *bounded* if for any map $\operatorname{Spec} A \to Y$, if we write $X \times_Y \operatorname{Spec} A = (\mathcal{X}, \mathcal{O})$, then the algebraic space $(\tau_{\leq 0} \mathcal{X}, \pi_0 \mathcal{O})$ is quasi-compact and quasi-separated. More generally, if $p : X \to Y$ is a relative stack, then we shall say that p is *bounded* if for each $\operatorname{Spec} A \to Y$, there exists a bounded smooth surjection $\operatorname{Spec} B \to X \times_Y \operatorname{Spec} A$.

We note that Definition 5.3.2 is recursive; in order to test whether or not $p : X \to Y$ is bounded, we need to know whether or not a smooth surjection $q : U \to X \times_Y \operatorname{Spec} A$ is bounded. However, this poses no difficulty, since if p is an n-stack, then q is an $(n-1)$-stack, so we eventually reduce to the case of a relative algebraic space. We note that the definition is consistent, since an algebraic space X is quasi-compact and quasi-separated if and only if there exists a smooth surjection $U \to X$ which is quasi-compact (and quasi-separated), where U is affine.

It is easy to give a characterization of the bounded derived schemes. For this, we first need a bit of terminology. Let $\mathfrak{X}$ denote the étale topos of an (ordinary) Deligne-Mumford stack X, and let E be a sheaf of spaces on $\mathfrak{X}$. We will say that E is *constructible* if it satisfies the following conditions:

- The sheaf of sets $\pi_0 E$ is constructible in the classical sense. That is, it is a compact object in the topos $\mathfrak{X}$.
- For any étale map $\operatorname{Spec} A \to X$ and any global section η of E over $\operatorname{Spec} A$, the homotopy sheaves $\pi_n(E|\operatorname{Spec} A, \eta)$ are constructible, and vanish for $n \gg 0$.

One can show that constructibility admits a recursive characterization analogous to that of Definition 5.3.2: an n-truncated E sheaf of spaces on X is constructible if and only if there is a constructible sheaf of sets E_0 and surjection $E_0 \to E$ whose homotopy fibers are all constructible. Since these homotopy fibers are $(n-1)$-truncated, we eventually reduce to the case where E is a sheaf of sets, in this case we apply the classical definition.

We shall say that a derived stack X is *bounded* if there exists a bounded submersion $\operatorname{Spec} A \to X$, or equivalently if $X \to \operatorname{Spec} \mathbf{Z}$ is bounded.

Proposition 5.3.3. *Let $X = (\mathcal{X}, \mathcal{O})$ be a derived scheme. Let $\mathcal{X} = (\Delta\tau_{\leq 0}\mathcal{X})_{/E}$. Then X is a bounded derived stack if and only if the Deligne-Mumford stack $(\tau_{\leq 0}\mathcal{X}, \pi_0\mathcal{O})$ is quasi-compact and quasi-separated (with quasi-separated diagonal), and the sheaf E is constructible.*

The following properties of bounded morphisms are easily verified:

Proposition 5.3.4. 1. *Any identity morphism is bounded. And morphism homotopic to a bounded morphism is bounded.*

2. *A composition of bounded morphisms is bounded. Any base change of a bounded morphism is bounded.*

3. *Let $X \to Y \to Z$ be a composable pair of relative stacks (or relative derived schemes). Assume that both $X \to Z$ and $Y \to Z$ are bounded. Then $X \to Y$ is bounded. In particular, if $X \to \operatorname{Spec} A$ is bounded, then* any *smooth surjection* $\operatorname{Spec} B \to X$ *is bounded.*

Now that we have introduced the class of bounded morphisms, we are in a position to set up the basic inductive apparatus for proving theorems about derived stacks:

Principle 5.3.5 (Unscrewing). *Let P be a property of objects $X \in \operatorname{Shv}(\mathcal{SCR}^{op})$. Consider the following conditions on P:*

1. *Every affine derived scheme has the property P.*

2. *If $U_0 \to X$ is surjective, and for each $k \geq 0$, the k-fold fiber power U_k of U_0 over X has the property P, then X has the property P.*

3. *If $\{X_\alpha \subseteq X\}$ is a filtered system of open subfunctors of X with union equal to X, and each X_α has the property P, then X has the property P.*

If (1) *and* (2) *are satisfied, then every bounded derived stack has the property P. If* (1), (2), *and* (3) *are satisfied, then any derived stack has the property P.*

Proof. First assume that (1) and (2). Let X be a bounded derived n-stack; we must show that X has the property P. We work by induction on n. If $n > 0$, then there exists an $(n-1)$-submersion $U_0 \to X$, where U_0 is affine. Let U_k denote the $(k+1)$-fold fiber power of U_0 over X; then each U_k is a bounded $(n-1)$-stack, so we may suppose that each U_k has the property P. Then (2) implies that p has the property P. Therefore it suffices to treat the case where $n = 0$, so that X is a derived algebraic space.

If X is affine, then (1) implies that p has the property P. Next suppose that X is separated. There exists an étale surjection $U_0 \to X$ where U_0 is affine. Since X is separated, each U_k is affine and affine over X. Consequently, each map $U_k \to X$ and $U_k \to Y$ has the property P, so that p has the property P by (2). Finally, in the general case, we again choose an étale surjection $U_0 \to X$ with U_0 affine. Then U_0 is separated and separated over X. One shows by induction that each U_k is separated over X and over Y, so that $U_k \to X$ and $U_k \to Y$ have the property P. It follows that U has the property P.

If, in addition, condition (3) is satisfied, then the same proof works for any stack. The only difference is that we cannot assume that the coverings U_0 are affine. We can, however, assume that U_0 is a disjoint union of affine schemes, and using condition (3) we may employ a limit argument to reduce to the affine case. □

Remark 5.3.6. We will rarely apply Principle 5.3.5 in precisely the form that it is stated. More often, we will be discussing relative stacks $X \to Y$. In this case, we need an additional condition: that a relative stack $X \to Y$ has the property P if and only if every base change $X \times_Y \operatorname{Spec} A \to \operatorname{Spec} A$ has the property P. Under this assumption (always satisfied in practice) and assumptions analogous to those of Principle 5.3.5, we may show that every relative derived stack (bounded relative derived stack) has the property P.

We can also use the argument of Principle 5.3.5 to show that certain properties imply others. If P is some property of simplicial commutative rings which is stable under smooth base change, and P' is some property of functors in $\mathcal{S}\mathrm{hv}(\mathcal{SCR}^{op})$, then the argument of Principle 5.3.5 can be used to show that every derived stack X having the property P locally also has the property P', provided that the conditions (2) and (3) hold for the property P, together with the following replacement for (1): If $A \in \mathcal{SCR}$ has the property P, then the derived stack $\operatorname{Spec} A$ has the property P'.

As our first application of Principle 5.3.5, we prove that derived stacks are infinitesimally cohesive functors:

Proposition 5.3.7. *Let $p : X \to Y$ be a relative derived stack. Then p is nilcomplete and infinitesimally cohesive.*

Proof. Without loss of generality, we may suppose that Y is an affine derived scheme. Then X is a derived stack, and we must show that X is nilcomplete and infinitesimally cohesive. Suppose that $A \in \mathcal{SCR}$ is the limit of some diagram $\{A_\alpha\}$ in $\mathcal{SCR}$. We further suppose that this limit has the form of either a tower $\{\tau_{\leq n} A\}$ or a fiber product $A \simeq B \otimes_{B \oplus M[1]} B$. We wish to show that the natural map $X(A) \to \lim\{X(A_\alpha)\}$ is an equivalence.

Let $\mathcal{Y}$ denote the underlying ∞-topos of $\operatorname{Spec} A$. We note that $\mathcal{Y}$ may also be identified with the underlying ∞-topos of each $\operatorname{Spec} A_\alpha$. For each $Z \in \operatorname{Shv}(\mathcal{SCR}^{op})$, we let $\mathcal{F}_Z$ denote the object of $\mathcal{Y}$ given by restricting Z to étale A-algebras, and $\mathcal{F}_Z^\alpha$ the object of $\mathcal{Y}$ given by restricting Z to étale A_α-algebras. Let $P(Z)$ be the assertion that the natural map $\mathcal{F}_Z \to \lim\{\mathcal{F}_Z^\alpha\}$ is an equivalence. We will complete the proof by applying Principle 5.3.5 to conclude that every derived stack has the property P.

It is obvious that P satisfies conditions (1) and (4) of Principle 5.3.5. Condition (2) follows from the fact that any open cover of $\mathcal{Y}$ has a finite subcover (in other words, the compactness of the ordinary Zariski spectrum of $\pi_0 A$). To verify (3), let us consider a submersion $U_0 \to X$, and let U_k denote the $(k+1)$-fold fiber power of U_0 over X. We must show that if each U_k has the property P, then X has the property P.

We note that $\lim\{\mathcal{F}_{U_k}^\alpha\}$ is the $(k+1)$-fold fiber power of $\lim\{\mathcal{F}_{U_0}^\alpha\}$ over $\lim\{\mathcal{F}_X^\alpha\}$. By hypothesis, these limits are simply given by $\mathcal{F}_{U_k}$, which is also the $(k+1)$-fold fiber power of $\mathcal{F}_{U_0}$ over $\mathcal{F}_X$. It follows that the natural map $f : \mathcal{F}_X \to \lim\{\mathcal{F}_X^\alpha\}$ is (-1)-truncated, and so it suffices to prove that f is surjective. Since the natural map $\mathcal{F}_{U_0}$ factors through f, it will suffice to show that the natural map $f' : \lim\{\mathcal{F}_{U_0}^\alpha\} \to \lim\{\mathcal{F}_X^\alpha\}$ is surjective.

In fact, we will show that f' induces a surjection when evaluated on any étale A-algebra A'. Replacing A by A', we reduce to proving that the natural map $\lim\{U_0(A_\alpha)\} \to \lim\{X(A_\alpha)\}$ is surjective.

Suppose first that A is given by a small extension $B \times_{B \oplus M[1]} B$. Suppose that A' is an étale A-algebra, and let $B' = B \otimes_A A'$, $M' = M \otimes_A A'$. We must show that any point of $X(B') \times_{X(B' \oplus M'[1])} X(B')$ lifts locally to U_0. Since $U_0 \to X$ is surjective, we may after localizing further suppose that the corresponding point of $X(B' \oplus M'[1])$ lifts to a point of $U_0(B' \oplus M'[1])$. Now it suffices to prove that both of the natural maps $U_0(B') \to U_0(B' \oplus M'[1]) \times_{X(B' \oplus M'[1])} X(B')$ are surjective. This follows from the fact that $L_{U_0/X}$ is the dual of a connective, perfect complex. This completes the proof of the assertion that every derived stack is infinitesimally cohesive.

Now suppose that the inverse system $\{A_\alpha\}$ is simply the tower $\{\tau_{\leq n} A\}$ of truncations of A. Suppose we are given a point of $\lim\{X(\tau_{\leq n} A)\}$ over some étale A-algebra A'. Shrinking A', we may suppose that the corresponding point in $X(\tau_{\leq 0} A')$ lifts to $U_0(\tau_{\leq 0} A')$. It now suffices to prove the surjectivity of the map

$$U_0(\tau_{\leq n+1} A') \to U_0(\tau_{\leq n} A') \times_{X(\tau_{\leq n} A')} X(\tau_{\leq n+1} A').$$

This follows from the fact that $\tau_{\leq n+1} A'$ is a small extension of $\tau_{\leq n} A'$, the assumption on $L_{U_0/X}$, and the first part of the proof. □

We will later show that every derived stack is cohesive (see Theorem 5.6.4).

Corollary 5.3.8. *If $n \geq 0$, $p : X \to Y$ is a relative n-stack, and $A \in \mathcal{SCR}$ is k-truncated, then the map $X(A) \to Y(A)$ is $(n+k)$-truncated (that is, has $(n+k)$-truncated homotopy fibers).*

Proof. Fix any point $\eta \in X(A)$. It suffices to show that the homotopy fiber of

$$X(\tau_{\leq j+1}A) \to X(\tau_{\leq j}A) \times_{Y(\tau_{\leq j}A)} Y(\tau_{\leq j+1}A)$$

is $(n+k)$-truncated whenever $j+1 \leq k$ (where the homotopy fiber is taken over the point induced by η). Since $\tau_{\leq j+1}A$ is a square-zero extension of $\tau_{\leq j}A$, the homotopy fiber is a torsor for

$$\mathrm{Hom}_{\mathcal{M}_A}(L_{X/Y}(\eta), (\pi_{j+1}A)[j+1])$$

which is $(j+1+n)$-truncated since $L_{X/Y}[n]$ is connective. □

Corollary 5.3.9. *Let X be a derived stack. Then X is a hypersheaf with respect to the étale topology.*

Proof. Since X is nilcomplete, X is given by the inverse limit of the functors $A \mapsto X(\tau_{\leq n}A)$. Corollary 5.3.8 implies that each of these functors is a truncated étale sheaf, hence an étale hypersheaf. Since X is an inverse limit of étale hypersheaves, it is itself an étale hypersheaf. □

As a second application of Principle 5.3.5, we discuss the functorial characterization of "almost of finite presentation".

Proposition 5.3.10. *Let $p : X \to Y$ be a relative stack. The morphism p is locally of finite presentation to order n if and only if the following condition is satisfied: for any $\eta \in Y(A)$, the functor $\tau_{\leq n}\,\mathcal{SCR}_{A/} \to \mathcal{S}$, given taking the fiber of p over η, commutes with filtered colimits.*

Proof. The assertion is local on Y, so we may suppose that $Y = \operatorname{Spec} A$ is affine. The proof of the "only if" direction uses the relative version of Principle 5.3.5: conditions (1) and (3) are immediate, while condition (2) follows from Theorem 4.4.3 and 4.4.4 by passing to the geometric realization.

For the "if" direction, we may reduce to the case where $Y = \operatorname{Spec} B$. Suppose that $X \to Y$ satisfies the condition. Any smooth morphism $\operatorname{Spec} A \to X$ is locally of finite presentation, and therefore also satisfies the condition. It follows that $\operatorname{Spec} A \to \operatorname{Spec} B$ satisfies the stated condition, so that A is of finite presentation over B to order n by definition. □

5.4 Truncated Stacks

The property of being an n-truncated object of $\mathcal{SCR}$ is local for the smooth topology (or even the flat topology). Consequently, we may speak of *n-truncated derived stacks*: a derived stack X is *n-truncated* if and only if there exists a submersion $U \to X$, where U is a derived scheme which is locally equivalent to $\operatorname{Spec} A$ for some n-truncated $A \in \mathcal{SCR}$. Equivalently, X is n-truncated if and only if A is n-truncated for any smooth relative stack $\operatorname{Spec} A \to X$. We note that this notion has no obvious relative analogue.

Definition 5.4.1. Let X be a derived stack. An *n-truncation* of X is a map $p : Y \to X$ of derived stacks, where Y is n-truncated and $X(A) \simeq Y(A)$ for any n-truncated $A \in \mathcal{SCR}$.

Remark 5.4.2. If $p : Y \to X$ induces an equivalence $Y(A) \simeq X(A)$ for any n-truncated $A \in \mathcal{SCR}$, then $\mathrm{Hom}(Z, Y) \to \mathrm{Hom}(Z, X)$ is an equivalence for any n-truncated derived stack Z (apply Proposition 5.3.5).

Remark 5.4.3. If $p : X \to Y$ is smooth and Y is n-truncated, then X is also n-truncated. The converse holds if p is surjective.

We first prove that n-truncations exist:

Proposition 5.4.4. *Let X be a derived stack. Then there exists an n-truncation for X.*

Proof. If $X = (\mathcal{X}, \mathcal{O})$ is a derived scheme, then we may take $Y = (\mathcal{X}, \tau_{\leq n}\mathcal{O})$. In order to handle the general case we apply Proposition 5.3.5. The only nontrivial point to verify is that if $U_0 \to X$ is a submersion, U_k the $(k+1)$-fold fiber power of U_0 over X, and each U_k has an n-truncation V_k, then X has an n-truncation. Since each $U_k \to U_0$ is smooth, the fiber product $U_k \times_{U_0} V_0$ is n-truncated and therefore equivalent to V_k.

Let $Y = |V_\bullet|$ as $\mathcal{T}$-sheaves on $\mathcal{SCR}^{op}$. One easily checks that $V_\bullet$ is a groupoid object and therefore effective, so that $V_1 \simeq V_0 \times_Y V_0$. Then $V_0 \to Y$ is surjective; it suffices to show that $V_0 \to Y$ is a submersion. Choose any map $\eta : \operatorname{Spec} A \to Y$; it suffices to show that $\operatorname{Spec} A \times_Y V_0$ is a relative stack smooth over $\operatorname{Spec} A$. The assertion is local on $\operatorname{Spec} A$, so we may assume that η factors through V_0. Then $\operatorname{Spec} A \times_Y V_0 \simeq \operatorname{Spec} A \times_{V_0} V_1 = \operatorname{Spec} A \times_{U_0} U_1$ is a relative stack smooth over $\operatorname{Spec} A$, as desired. □

If X is a derived stack, we shall denote its n-truncation by $\tau_{\leq n}X$. We have a natural directed system

$$\tau_{\leq 0}X \to \tau_{\leq 1}X \to \dots$$

Proposition 5.4.5. *Let X and Y be derived stacks. Then* $\mathrm{Hom}(X, Y)$ *is equivalent to the limit of the tower* $\{\mathrm{Hom}(\tau_{\leq n}X, Y)\} = \{\mathrm{Hom}(\tau_{\leq n}X, \tau_{\leq n}Y)\}$.

Proof. By applying Principle 5.3.5 to X, we may reduce to the case where X is affine. In this case, the assertion follows from the fact that Y is infinitesimally cohesive. □

We may interpret Proposition 5.4.5 as asserting that a derived stack X may be recovered from its truncations $X_n = \tau_{\leq n}X$. The following result shows that there are essentially no restrictions on the X_n other than the obvious ones:

Proposition 5.4.6. *Suppose given a sequence $\{X_0, X_1, \dots,\}$ of derived stacks, and equivalences $X_i \simeq \tau_{\leq i}X_{i+1}$ (so that each X_i is i-truncated). Then there exists a derived stack X and a coherent family of equivalences $X_i \simeq \tau_{\leq i}X$.*

Proof. If A is n-truncated, we let $X(A)$ denote the direct limit of the sequence of spaces $\{X_m(A)\}$. We note that the sequence in question is constant for $m \geq n$. For general A, we let $X(A)$ be the inverse limit of the spaces $X(\tau_{\leq n}A)$. It is clear that X_n and X agree on n-truncated objects. Since X_n is n-truncated, this furnishes a family of equivalences

$X_n \simeq \tau_{\leq n} X$ which are easily seen to be compatible with the equivalences $X_n \simeq \tau_{\leq n} X_{n+1}$. To complete the proof, it suffices to show that X is a derived stack. The assertion is local on X_0, so we may suppose that X_0 is a derived k-stack. We prove the result by induction on k.

First suppose that $k = 0$, so that $X_0 = (\mathcal{X}, \mathcal{O}_0)$ is a derived scheme. One can then check that each X_i is representable by a derived scheme $(\mathcal{X}, \mathcal{O}_i)$ having the same underlying ∞-topos. One may then take X to be the derived scheme $(\mathcal{X}, \mathcal{O}_\infty)$, where $\mathcal{O}_\infty$ is the inverse limit of the sequence $\{\mathcal{O}_i\}$.

If $k > 0$, then there exists a submersion $p_0 : U_0 \to X_0$ where U_0 is a disjoint union of affine derived schemes. Now we claim that U_0 admits a thickening to a 1-truncated derived scheme U_1 which is smooth over X_1, and such that $U_0 \simeq \tau_{\leq 0} U_1$. The obstruction to the existence of this thickening lies in $\pi_{-1} \operatorname{Hom}_{\mathrm{QC}_{U_0}}(L_{U_0/X_0}, p_0^* M)$, where M is the quasi-coherent complex on X_0 given by the difference between the structure sheaves of X_1 and X_0. This group vanishes since M is connective (even 0-connected), L_{U_0/X_0} is locally free, and U_0 is a disjoint union of affine schemes.

Iterating this argument, we obtain an inverse system $\{U_i\}$ of infinitesimal thickenings of U_0. Since U_0 is a derived scheme, we may construct the direct limit of these thickenings U as a derived scheme. To complete the proof, it suffices to show that $U \to X$ is a $(k-1)$-submersion. It is easy to show that $U \to X$ is surjective, so it suffices to prove this result after base change to U. In other words, we must show that $U \times_X U \to U$ is a $(k-1)$-submersion. This follows from the inductive hypothesis. □

Corollary 5.4.7. *Let $X \to \operatorname{Spec} B$ be a morphism in $\operatorname{Shv}(\mathcal{SCR}^{op})$, and suppose that X is infinitesimally cohesive. Suppose further that each $X_n = X \times_{\operatorname{Spec} B} \operatorname{Spec}(\tau_{\leq n} B)$ is a derived stack. Then X is a derived stack. Moreover, if each X_n is almost of finite presentation over $\tau_{\leq n} B$, then X is almost of finite presentation over B.*

Proof. Let $X'_n = \tau_{\leq n} X_n$. We note that X, X_n, and X'_n have the same values on any n-truncated object of $\mathcal{SCR}$. Consequently, there are natural equivalences $X'_n \simeq \tau_{\leq n} X'_{n+1}$, so by Proposition 5.4.6 we can glue together the X'_n to make a derived stack X'. By construction, X' and X define the same functor on truncated objects of $\mathcal{SCR}$. Since $X(A) \simeq \lim\{X(\tau_{\leq n} A)\} = \lim\{X'(\tau_{\leq n} A)\} \simeq X'(A)$, we deduce that X and X' are equivalent so that X is a derived stack.

To prove the last claim, we may work locally on X and therefore assume that $X = \operatorname{Spec} A$. To show that A is almost of finite presentation over B, it suffices to show that A is of finite presentation over B to order n for each $n \geq 0$. This follows from the fact that $A \otimes_B (\tau_{\leq n} B)$ is of finite presentation over $\tau_{\leq n} B$ to order n. □

We now formulate another sense in which a derived stack X behaves like the direct limit of its truncations $\tau_{\leq n} X$. Let $\mathrm{QC}_{\hat{X}}$ denote the (strict) inverse limit of the ∞-categories $\mathrm{QC}_{\tau_{\leq n} X}$. Then there is a natural restriction map $\phi : \mathrm{QC}_X \to \mathrm{QC}_{\hat{X}}$, and a "completion" functor ψ which is right adjoint to ϕ (given by forming inverse limits). It is not always the case that ϕ and ψ are inverse equivalences. However, we can assert the existence of such an equivalence for *connective* complexes. Here we say that an object $\{M_n\} \in \mathrm{QC}_{\hat{X}}$ is *connective* if each $M_n \in \mathrm{QC}_{\tau_{\leq n} X}$ is connective.

Proposition 5.4.8. *The functors $\phi : \mathrm{QC}_X \to \mathrm{QC}_{\hat{X}}$ and $\psi : \mathrm{QC}_{\hat{X}} \to \mathrm{QC}_X$ carry connective objects into connective objects, and the adjunction morphisms $M \to \psi\phi M$ and $\phi\psi N \to N$ are equivalences whenever M and N are connective.*

Proof. The assertion is local so we may reduce to the case where $X = \operatorname{Spec} A$ for $A \in \mathcal{SCR}$. Let M be an A-module. Then $\psi\phi M = \lim\{M \otimes_A \tau_{\leq n} A\}$. If M is connective, then $\pi_i M \simeq \pi_i(M \otimes_A \tau_{\leq n} A)$ for $i \leq n$. Passing to the limit, we deduce that $M \simeq \psi\phi M$.

Now suppose that $N = \{N_i\} \in \mathrm{QC}_{\hat{X}}$ is given by a compatible family of $\tau_{\leq i} A$-modules N_i. Then ψN is given by the inverse limit $\lim\{N_i\}$, so that we have for each k a short exact sequence

$$0 \to \lim{}^1\{\pi_{k+1} N_i\} \to \pi_k \psi N \to \lim{}^0\{\pi_k N_i\} \to 0.$$

If each N_i is connective, then this exact sequence shows that $\pi_i(\psi N)$ vanishes for $i < -1$. Moreover, the sequence $\pi_0 N_i$ is constant, so the corresponding $\lim^1$-term vanishes and we get also $\pi_{-1}(\psi N) = 0$. Thus ψN is connective.

To show that the adjunction $\phi\psi N \to N$ is an equivalence, it suffices to show that its cokernel K vanishes. Since $\psi\phi$ is equivalent to the identity, we deduce that $\psi K = 0$. If N is connective, then K is connective. Choose n minimal such that $\pi_n K_0 \neq 0$. Then one deduces that the sequence $\{\pi_n K_i\}$ is constant, and makes a nonzero contribution to $\pi_n(\psi K)$, a contradiction. □

We conclude this section by setting up a framework for direct limit arguments. If $A \in \mathcal{SCR}$ is the colimit of a filtered system $\{A_\alpha\}$, then one would like to say that the theory of derived stacks over $\operatorname{Spec} A$ may be obtained as a sort of direct limit of the theories of derived schemes over the $\operatorname{Spec} A_\alpha$. Of course, this is not true in complete generality, but requires some finiteness conditions on the derived schemes in question. The most natural finiteness condition would assert that the derived schemes in question are locally of finite presentation over $\operatorname{Spec} A$. However, this condition will turn out to be too restrictive for many purposes. On the other hand, if X is almost of finite presentation over $\operatorname{Spec} A$, then X need not arise as the base change of some derived scheme X_α over some A_α. However, we will show that we can often *approximate* X by derived schemes over some A_α, and this will be sufficient for our later purposes.

The following result has essentially already been proven:

Proposition 5.4.9. *Let $A_0 \in \mathcal{SCR}$ and let $\{A_\alpha\}$ be a filtered system of A_0-algebras having colimit A. Let X_0 and Y_0 be derived stacks over A, let $X_\alpha = X_0 \times_{\operatorname{Spec} A_0} \operatorname{Spec} A_\alpha$, $Y_\alpha = Y_0 \times_{\operatorname{Spec} A_0} \operatorname{Spec}_{A_\alpha}$, $X = X_0 \times_{\operatorname{Spec} A_0} \operatorname{Spec} A$ and $Y = Y_0 \times_{\operatorname{Spec} A_0} \operatorname{Spec} A$.*

If X_0 is bounded and Y_0 is locally of finite presentation to order n over $\operatorname{Spec} A_0$, then

$$\operatorname{colim} \operatorname{Hom}_{\operatorname{Spec} A_\alpha}(\tau_{\leq n} X_\alpha, \tau_{\leq n} Y_\alpha) \to \operatorname{Hom}_{\operatorname{Spec} A}(\tau_{\leq n} X, \tau_{\leq n} Y)$$

is an equivalence.

Proof. Using Principle 5.3.5, we may reduce to the case where X is affine. Then the result follows immediately from Proposition 5.3.10. □

It remains to show that, under suitable finiteness conditions, any derived stack X over $\operatorname{Spec} A$ may be obtained as the base change of a derived stack over some $\operatorname{Spec} A_\alpha$:

Proposition 5.4.10. *Let $\{A_\alpha\}$ be a filtered system in $\mathcal{SCR}$ having colimit A. Suppose that X is a derived stack which is bounded, n-truncated, and locally of finite presentation to order n over $\operatorname{Spec} A$. Then there exists an index α, a derived stack X_α which is bounded, n-truncated, and locally of finite presentation to order n over $\operatorname{Spec} A_\alpha$, and an equivalence $X \simeq \tau_{\leq n}(X_\alpha \times_{\operatorname{Spec} A_\alpha} \operatorname{Spec} A)$.*

Proof. We apply Principle 5.3.5. Suppose first that $X = \operatorname{Spec} B$ is affine. Then $B = \tau_{\leq n} B'$, where B' is of finite presentation over A. Lifting cell-by-cell, we may construct (for sufficiently large α) an A_α-algebra B'_α together with an equivalence $B'_\alpha \otimes_{A_\alpha} A$. Then we may take $X_\alpha = \operatorname{Spec}(\tau_{\leq n} B'_\alpha)$.

To complete the proof, let us suppose that X is a derived stack which is bounded, n-truncated, and locally of finite presentation to order n over $\operatorname{Spec} A$, and that $U_0 \to X$ is a submersion with U_0 affine. Let U_k denote the $(k+1)$-fold fiber power of U_0 over X; we may suppose that the theorem is known for each U_k. Choose $m \gg 0$ such that X is an m-stack; then $X(B)$ is $(n+m)$-truncated whenever B is n-truncated. It follows that for $k > n+m$, $\tau_{\leq n} U_k$ is the k-truncation of the kth space of the $(k-1)$-coskeleton of $U_\bullet$. Since U_k is smooth over X and X is n-truncated, we deduce that $U_\bullet$ is $(n+m)$-coskeletal. Consequently, if α is sufficiently large, then $U_\bullet$ is the n-truncation of the base change of an $(n+m)$-coskeletal, n-truncated simplicial stack $(U_\alpha)_\bullet$ over $\operatorname{Spec} A_\alpha$. Enlarging α if necessary, we may ensure that $(U_\alpha)_\bullet$ is a smooth groupoid, and then we may take X_α to be its geometric realization. □

Of course, it is possible to prove many similar and related results. In particular, in the next section we will need to know that if $M \in \mathrm{QC}_X$ is n-truncated and perfect to order n, then for α sufficiently large there exists $M_\alpha \in \mathrm{QC}_{X_\alpha}$, which is perfect to order n, and an equivalence $M \simeq \tau_{\leq n}(M_\alpha | X)$. The proof may be given along the same lines as that of Proposition 5.4.10.

5.5 Coherence Theorems

Throughout this section, $\mathcal{T}$ shall denote the étale topology.

Definition 5.5.1. A morphism $p : X \to Y$ in $\operatorname{Shv}(\mathcal{SCR}^{op})$ is *proper* if it is a relative derived scheme, almost of finite presentation, and for any morphism $\operatorname{Spec} A \to Y$, the fiber product $X \times_Y \operatorname{Spec} A \simeq (\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$, where $\mathcal{Z} \simeq \Delta \tau_{\leq 0} \mathcal{Z}$ and the Deligne-Mumford stack $(\tau_{\leq 0} \mathcal{Z}, \pi_0 \mathcal{O}_{\mathcal{Z}})$ is proper over $\pi_0 A$ in the usual sense.

Remark 5.5.2. Our definition of proper morphisms is slightly more restrictive than the standard definition because we require proper morphisms to be almost of finite presentation. This disallows, for example, closed immersions for which the ideal sheaf is not locally finitely generated. Otherwise, our definition is the obvious derived analogue of the usual notion of a proper morphism.

Remark 5.5.3. If $p : X \to Y$ is a morphism between derived stacks which is almost of finite presentation, then the condition that p be proper is a purely "topological" notion. That is, p is proper if and only if the induced map $\tau_{\leq 0}X \to \tau_{\leq 0}Y$ is proper, so that the condition of properness is insensitive to the higher homotopy groups of the structure sheaves.

We next study the pushforward functor on quasi-coherent complexes. Suppose that $p : X \to Y$ is a relative stack. If $Y : \mathcal{SCR}^{op} \to \mathcal{S}$, then so is X. Then both QC_Y and QC_X are presentable ∞-categories. The restriction functor $p^* : \mathrm{QC}_Y \to \mathrm{QC}_X$ commutes with all colimits, and therefore has a right adjoint by the adjoint functor theorem. We shall denote this adjoint by p_*. In general, the functor p_* may be very poorly behaved. However, if p is a relative derived algebraic space, then we can say a great deal about p_*. First, we need a lemma.

Lemma 5.5.4. *Let X be an (ordinary) algebraic space which is quasi-compact and separated. Then there exists $n \geq 0$ such that for any quasi-coherent sheaf M on X, $\mathrm{H}^m(X, M) = 0$ for $m > n$.*

Proof. Choose an affine scheme U and an étale morphism $p : U \to X$. We may take n to be an upper bound for the number of points in the geometric fibers of p (to see this, compute the cohomology using strictly alternating cochains). □

Proposition 5.5.5. *Let $p : X \to Y$ be a bounded, separated, relative derived algebraic space.*

1. *The functor p_* exists and commutes with all colimits.*
2. *Let $q : Y' \to Y$ be any morphism, let $X' = X \times_Y Y'$, and let $p' : X' \to Y'$ and $q' : X' \to X$ be the induced maps. The natural base change morphism $q^* p_* \to p'_*(q')^*$ is an equivalence of functors $\mathrm{QC}_X \to \mathrm{QC}_{Y'}$.*
3. *If p is affine, then p_* carries connective complexes into connective complexes.*
4. *If p is of* Tor*-amplitude $\leq k$, then p_* carries complexes of* Tor*-amplitude $\leq n$ into complexes of* Tor*-amplitude $\leq (n + k)$.*
5. *If p is proper, then p_* carries almost perfect complexes into almost perfect complexes.*
6. *If p is proper and flat, then p_* carries perfect complexes into perfect complexes.*

Proof. First suppose that $Y = \operatorname{Spec} A$ is affine. In this case, $X = (\mathfrak{X}, \mathcal{O})$ is a bounded derived algebraic space, and p_* exists by the adjoint functor theorem. We may identify QC_Y with $\mathcal{M}_A$ and QC_X with a full subcategory of $\mathcal{M}_{\mathcal{O}}$. In this case, p_* is simply given by the global sections functor. If M is a *discrete* $\mathcal{O}$-module, then the homotopy groups of p_*M are simply the cohomology groups of M, regarded as a quasi-coherent sheaf on the underlying ordinary algebraic space of X. Since X is bounded, this algebraic space is quasi-compact and quasi-separated, so Lemma 5.5.4 implies that there exists n such that $\pi_m p_* M = 0$ for $m < n$.

By induction, one shows that if $M \in \mathrm{QC}_X$ has homotopy groups only in the range $[a, a+1, \ldots, b-1, b]$, then p_* has homotopy groups only in the range $[a-n, a-n+1 \ldots, b-1, b]$. Since p_* commutes with inverse limits, we have $p_* M = \lim\{p_*(\tau_{\leq k} M)\}$. Moreover, for each m the sequence of homotopy groups $\{\pi_m(p_* \tau_{\leq k} M)\}$ is constant for $k > m+n$. Thus all $\lim^1$-terms vanish and we obtain isomorphisms $\pi_m p_* M \simeq \pi_m(p_* \tau_{\leq k} M)$ for $k > n+m$. In particular, if M is j-connected, then $p_* M$ is $(j-n)$-connected.

To prove that p_* commutes with all colimits, it suffices to show that p_* commutes with direct sums, and for this it suffices to show that $\pi_m p_*$ commutes with direct sums. Since $\pi_m p_* M$ depends only on $\tau_{\leq m+n, \geq m} M$, it suffices to prove that p_* commutes with direct sums on $\tau_{\leq m+n, \geq m} \mathrm{QC}_X$. To prove this, we choose a submersion $U_0 \rightarrow X$, where U_0 is affine, and let U_k denote the $(k+1)$-fold fiber power of U_0 over X. Then there exists a spectral sequence with E_1-term given by $\pi_a M | U_b \Rightarrow \pi_{p+q} p_* M$. Since only finitely many terms in this spectral sequence can contribute to a particular homotopy group of $p_* M$, and the formation of the spectral sequence is compatible arbitrary direct sums in M, the desired result follows.

One can always construct a push-pull morphism $(p_* M) \otimes N \rightarrow p_*(M \otimes p^* N)$. If $N = A$, then this morphism is an equivalence. Since p_* commutes with colimits, we deduce that the push-pull morphism is always an equivalence. In the special case where $A \rightarrow B$ is a morphism in $\mathcal{SCR}$ and we take $N = B$ as an A-module, we deduce that (2) holds when $Y' \rightarrow Y$ is a transformation of affine derived schemes.

Now suppose that $p : X \rightarrow Y$ is arbitrary, and that $M \in \mathrm{QC}_X$. Define $p_+ M \in \mathrm{QC}_Y$ by the equation $p_+ M(\eta) = p'_* M | X'$, where $\eta : \operatorname{Spec} A \rightarrow Y$ is any morphism, $X' = X \times_Y \operatorname{Spec} A$, and $p' : X' \rightarrow \operatorname{Spec} A$ is the projection. The fact that the push-pull morphism is an equivalence implies that $p_+ M$ is compatible with base change, and therefore gives a well-defined quasi-coherent complex on Y. It is then easy to see that $p_+ M$ has the correct universal mapping properties, so that $p_* M = p_+ M$ exists. Moreover, this construction shows that p_* commutes with filtered colimits and base change in general. This proves (1) and (2). Part (3) may be reduced to the affine case, where it is obvious.

Now suppose that p is of Tor-amplitude $\leq k$ and that $M \in \mathrm{QC}_X$ is of Tor-amplitude $\leq n$. If $N \in \mathrm{QC}_Y$ is discrete, then $p^* N$ is k-truncated, so that $M \otimes p^* N$ is $(n+k)$-truncated. Thus $p_*(M \otimes p^* N) = (p_* M) \otimes N$ is $(n+k)$-truncated. This proves (4).

To prove (5), we first reduce to the case $Y = \operatorname{Spec} A$ is affine. Choose n as above so that $\pi_m p_* M$ depends only on $\pi_k M$ for $k \leq m+n$. We wish to show that if $M \in \mathrm{QC}_X$ is almost perfect, then $p_* M$ is almost perfect. Since X is bounded, M is almost connective; by shifting we may suppose that M is connective. It suffices to show that $p_* M$ is perfect to order m for each $m \geq 0$. We note that this condition depends only on $\pi_k M$ for $k \leq m+n$. We may therefore apply Proposition 5.4.10 (and the comments that follow it) to obtain $A_0 \in \mathcal{SCR}$ which is of finite presentation over $\mathbf{Z}$, a derived algebraic space X_0 which is $(m+n)$-truncated and locally of finite presentation to order $(m+n)$ over $\operatorname{Spec} A_0$, and a connective quasi-coherent complex $M_0 \in \mathrm{QC}_{X_0}$ which is $(m+n)$-truncated and perfect to order $(m+n)$, with identifications of the $(k+n)$-truncations of the pair (X, M) with the $(k+n)$-truncations of the pair $(X_0 \otimes_{\operatorname{Spec} A_0} \operatorname{Spec} A, M_0 | X)$. We note that since A_0 is Noetherian, X_0 is almost of finite presentation over A_0. Enlarging A_0 if necessary, we may suppose that X_0 is proper over $\operatorname{Spec} A_0$. We may therefore replace A by A_0 and X by X_0, thereby reducing to the case where A is Noetherian and M is truncated. Using the appropriate exact triangles, one can reduce to the case where M is discrete. The assertion that $p_* M$ is almost perfect is equivalent to the assertion that each cohomology group of M (considered as a quasi-coherent sheaf on the underlying algebraic space of X) is finitely presented as an A-module. This follows from the classical coherence theorem for proper direct images of coherent sheaves.

Finally, (6) follows immediately from (4) and (5). □

Of course, the condition that $p : X \to Y$ be a separated relative derived algebraic space is very strong. Consideration of p_* for more general morphisms p seems to raise delicate issues involving the commutation of limits and colimits. In order to avoid these issues, we will restrict our attention to *truncated* quasi-coherent complexes.

Proposition 5.5.6. *Let $p : X \to Y$ be a bounded morphism between derived stacks. Then:*

1. *The functor p_* exists, and commutes with filtered colimits when restricted to $\tau_{\leq 0} \mathrm{QC}_X$.*

2. *When restricted to truncated complexes, the formation of p_* is compatible with base change by any relative stack $Y' \to Y$ of finite* Tor*-amplitude.*

3. *For any truncated $M \in \mathrm{QC}_X$ and any $N \in \mathrm{QC}_Y$ of finite* Tor*-amplitude, the push-pull morphism $p_* M \otimes N \to p_*(M \otimes p^* N)$ is an equivalence.*

Proof. The existence of p_* follows from the adjoint functor theorem. To prove the rest, we first suppose that $Y = \operatorname{Spec} A$ is affine. We then apply Principle 5.3.5 to the derived stack X. If X is affine, then (1), (2) and (3) are obvious (and require no truncatedness or Tor-amplitude assumptions). Now suppose that $U_0 \to X$ is a submersion, U_k is the $(k+1)$-fold fiber power of U_0 over X, and the result is known for each $q_k : U_k \to Y$. If $M \in \mathrm{QC}_X$, then $p_* M$ is the geometric realization of the cosimplicial A-module $(q_\bullet)_*(M|U_\bullet)$. Then (1) follows from the spectral sequence for computing $\pi_n p_* M$, with E^1_{ab}-term given by $\pi_a (q_{-b})_* M|U_{-b}$, which is compatible with filtered colimits and contains only finitely many pieces which contribute to a given π_n. To prove (3), we write N as the filtered colimit of a system of finitely presented A-modules N_α. If N has Tor-amplitude $\leq a$ and M is b-truncated, then $M \otimes p^* N \simeq \operatorname{colim} \tau_{\leq a+b} M \otimes N_\alpha$. Using (1), we may reduce to the case where N is finitely presented. Using various exact triangles we may then reduce to the case where $N = A$, which is obvious. We note that if Y' is affine over Y, then (2) is really a special case of (3).

Now suppose that Y is an arbitrary derived stack. If $\eta : \operatorname{Spec} A \to Y$ is smooth, let us define $(p_+ M)(\eta)$ to be the A-module $p'_* M|X'$, where $X' = X \times_Y \operatorname{Spec} A$ and $p' : X \to \operatorname{Spec} A$ is the projection. If U and U' a derived schemes which are smooth over Y, then any map $U \to U'$ over Y is quasi-smooth, and therefore of finite Tor-amplitude. It follows that if M is truncated, then $p_+ M$ is compatible with base change, and therefore gives a well-defined object in $\mathcal{M}_Y \simeq \mathrm{QC}_Y$. When regarded as a quasi-coherent complex on Y, $p_+ M$

has the appropriate universal property and is therefore naturally equivalent to $p_* M$. This construction of p_* allows us to reduce the proofs of (1), (2) and (3) to the affine case which was handled above. □

We next prove a slightly different version of the coherence theorem, which seems to require working in a Noetherian setting.

Proposition 5.5.7. *Let Y be a Noetherian derived stack, and let $p : X \to Y$ be a proper morphism. If $M \in \mathrm{QC}_X$ is truncated and coherent, then $p_* M \in \mathrm{QC}_Y$ is truncated and coherent.*

Proof. Without loss of generality we may suppose $Y = \operatorname{Spec} A$ is affine, where $A \in \mathcal{SCR}$ is Noetherian. We note that $\pi_i p_* M$ does not change when we replace M by $\tau_{\geq i} M$. Thus, we may suppose that M has only finitely many nonvanishing homotopy groups. An induction reduces us to the case where M has only a single nonvanishing homotopy group. By shifting, we may suppose that M is discrete. In this case, M may be regarded as a coherent sheaf on the underlying Deligne-Mumford stack of X, which is proper over $\operatorname{Spec} \pi_0 A$. The $\pi_n p_* M$ are simply given by the cohomology groups of this coherent sheaf. The classical coherence theorem for proper direct images (see [20] for a proof in the context of proper Deligne-Mumford stacks) implies that these modules are finitely generated over $\pi_0 A$, as desired. □

As a corollary, we may deduce the following result which will be needed later:

Corollary 5.5.8. *Let $p : X \to \operatorname{Spec} A$ be a proper, flat, relative algebraic space, and let $M \in \mathrm{QC}_X$ be almost perfect. Then there exists an almost perfect A-module M' equipped with a morphism $M \to p^* M'$ which induces an equivalence*

$$\mathrm{Hom}_{\mathcal{M}_A}(M', N) \to \mathrm{Hom}_{\mathrm{QC}_X}(M, p^* N)$$

for every A-module N.

Proof. To prove that M' exists, it suffices to prove the existence of $\tau_{\leq n} M'$ having the universal property

$$\mathrm{Hom}_{\mathcal{M}_A}(\tau_{\leq n} M', N) \simeq \mathrm{Hom}_{\mathrm{QC}_X}(M, p^* N)$$

whenever N is n-truncated; we can then obtain M' by taking an inverse limit. We note that if N is n-truncated, then so is $p^* N$ (since p is flat), so that the space on the right depends only on $\tau_{\leq n} M$. Consequently, we may employ a direct limit argument to reduce to the case where A is of finite presentation over $\mathbf{Z}$. In this case, A has a dualizing module. We now simply apply Theorem 3.6.9 to the functor $N \mapsto \mathrm{Hom}_{\mathrm{QC}_X}(M, p^* N)$. □

5.6 Gluing along Closed Subschemes

Since the ∞-category $\mathcal{SCR}$ is a presentable, it has arbitrary limits; in particular, we may construct fiber products $A \times_C B$. Note that $\operatorname{Spec}(A \times_C B)$ is a pushout $\operatorname{Spec} A \coprod_{\operatorname{Spec} C} \operatorname{Spec} B$

in the ∞-category of *affine* derived schemes. However, the fiber product construction in $\mathcal{SCR}$ has extremely poor behavior from an algebraic point of view (even for ordinary commutative rings), so there is very little else that can be said about $\mathrm{Spec}(A \times_C B)$ in general. However, if $A \to C$ and $B \to C$ are both surjective morphisms, then the fiber product $A \times_C B$ is well behaved and we can say a great deal.

We first consider the behavior of modules over $R = A \times_C B$. We let $\mathcal{M}_A \times_{\mathcal{M}_C} \mathcal{M}_B$ denote the (strict) fiber product of the ∞-categories $\mathcal{M}_A$ and $\mathcal{M}_B$ over $\mathcal{M}_C$. In other words, an object of $\mathcal{M}_A \times_{\mathcal{M}_C} \mathcal{M}_B$ is a triple (M_A, M_B, h) where $M_A \in \mathcal{M}_A$, $M_B \in \mathcal{M}_B$, and h is an equivalence between $M_A \otimes_A C$ and $M_B \otimes_B C$. There is an obvious functor

$$\phi : \mathcal{M}_R \to \mathcal{M}_A \times_{\mathcal{M}_C} \mathcal{M}_B$$

which may be described (on objects) by the formula $\phi(M) = (M \otimes_R A, M \otimes_R B, h)$, where h is the natural equivalence $(M \otimes_R A) \otimes_A C \simeq M \otimes_R C \simeq (M \otimes_R B) \otimes_B C$. Moreover, ϕ has a right adjoint ψ which may be described by the formula

$$\psi(M_A, M_B, h) = M_A \times_{M_C} M_B,$$

where $M_C = M_A \otimes_A C \simeq M_B \otimes_B C$.

Suppose that $A \to C$ and $B \to C$ are surjective morphisms in $\mathcal{SCR}$. Geometrically, these morphisms correspond to closed immersions, and $A \times_C B \in \mathcal{SCR}$ is the "affine ring" of functions on the space which is obtained by gluing the spectra of A and B along the closed subset corresponding to the spectrum of C. We would like to assert that ϕ and ψ are inverse equivalences in this case. Unfortunately, this is not true in complete generality.

Example 5.6.1. Let $A = k[x]$, $B = k[y]$, and $C = k$, where k is a field and the morphisms $A \to C$ and $B \to C$ are given by sending x and y to zero. Let M_A denote the A-module which is the direct sum of copies of $C[i]$ for i odd, and let M_B be the B-module which is the direct sum of copies of $C[i]$ for i even. Then $M_A \otimes_A C$ and $M_B \otimes_B C$ are C-modules whose homotopy groups are 1-dimensional in each degree. Choose any equivalence $M_A \otimes_A C \simeq M_B \otimes_B C \simeq M_C$. One easily checks that the fiber product $M_A \times_{M_C} M_B$ is zero as an $A \times_C B$-module. Thus, ψ is not faithful.

However, we can assert an equivalence which is valid for *connective* modules.

Proposition 5.6.2. *Suppose that $A \to C$ and $B \to C$ are surjective morphisms in $\mathcal{SCR}$ with fiber product $R = A \times_C B$. The natural functor*

$$\phi : \mathcal{M}_R \to \mathcal{M}_A \times_{\mathcal{M}_C} \mathcal{M}_B$$

is fully faithful and induces an equivalence

$$(\mathcal{M}_R)_{\geq 0} \to (\mathcal{M}_A)_{\geq 0} \times_{\mathcal{M}_C} (\mathcal{M}_B)_{\geq 0}.$$

Proof. Given any object $(M_A, M_B, h) \in \mathcal{M}_A \times_{\mathcal{M}_C} \mathcal{M}_B$, we will write M_C for $M_A \otimes_A C$ (which is equivalent $M_B \otimes_B C$ via h). We let ψ denote the right adjoint to ϕ described above. The existence of ψ implies that ϕ commutes with all colimits. To prove that $\mathrm{Hom}_{\mathcal{M}_R}(M, N) \simeq \mathrm{Hom}(\phi M, \phi N)$, it suffices to treat the case where $M = R[j]$. In other words, we need only show that $N \simeq (N \otimes_R A) \times_{N \otimes_R C} (N \otimes_R B)$. This follows by tensoring the exact triangle

$$R \to A \oplus B \to C$$

by N.

We may use ϕ to identify $\mathcal{M}_R$ with a full subcategory of $\mathcal{M}_A \times_{\mathcal{M}_C} \mathcal{M}_B$. Suppose that $M = (M_A, M_B, h) \in \mathcal{M}_A \times_{\mathcal{M}_C} \mathcal{M}_B$. We shall call M *connective* if both M_A and M_B are connective. To complete the proof, we need to show that every connective object belongs to the essential image of ϕ. In other words, we wish to show that if M is connective, then the cokernel K of the adjunction map $\phi\psi M \to M$ is zero. Since M is connective, so is ψM, and therefore so is $\phi\psi M$; thus K is connective. On the other hand, $\psi K = 0$. Let $K = (K_A, K_B, g)$. If $K \neq 0$, then there exists some least value of $n \geq 0$ such that $K[-n]$ is connective and $\pi_n K_A \oplus \pi_n K_B \neq 0$. In this case, there exists a short exact sequence

$$0 = \pi_n \psi K \to \pi_n K_A \times \pi_n K_B \to \pi_n K_C \to 0.$$

On the other hand, $\pi_n K_C = \mathrm{Tor}_0^{\pi_0 A}(\pi_n K_A, \pi_0 C) = \mathrm{Tor}_0^{\pi_0 B}(\pi_n K_B, \pi_0 C)$, so that $\pi_n K_C \simeq \pi_n K_A \oplus \pi_n K_B$. Since $\pi_0 A \to \pi_0 C$ and $\pi_0 B \to \pi_0 C$ are both surjective, the maps $\pi_n K_A \to \pi_n K_C$ and $\pi_n K_B \to \pi_n K_C$ are surjective. This implies that $\pi_n K_A = \pi_n K_B = 0$, a contradiction. □

Our next goal is to show that (assuming the appropriate surjectivity conditions) $\mathrm{Spec}(A \times_C B)$ is given by the pushout $\mathrm{Spec}\, A \coprod_{\mathrm{Spec}\, C} \mathrm{Spec}\, B$ in a larger ∞-category of derived stacks. It will be convenient to state and prove this result in a relative form; for this we need a bit of notation. Let $p : X \to Y$ be a map in $\mathrm{Shv}(\mathcal{SCR}^{op})$. If $\eta \in Y(R)$, we let $\mathcal{F}^p_\eta$ denote the homotopy fiber of $X(R) \to Y(R)$ over η, considered as a sheaf on $\mathrm{Spec}\, R$.

Lemma 5.6.3. *Let $p : X \to Y$ be a smooth surjection and a relative derived stack, let $f : A \to B$ be a surjective morphism in $\mathcal{SCR}$, and let $F : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ denote the corresponding closed immersion of derived schemes. Choose any $\eta \in Y(A)$, and let η' denote the restriction of η to $Y(B)$. Then the natural map $\mathcal{F}^p_\eta \to F_* \mathcal{F}^p_{\eta'}$ is a surjection of étale sheaves on* $\mathrm{Spec}\, A$.

Proof. Without loss of generality, we may replace Y by $\mathrm{Spec}\, A$. The surjectivity of $X \to Y$ remains valid for some $X_{(n)}$, so we may suppose that X is a relative n-stack over Y. Then there exists an $(n-1)$-submersion $U \to X$, where U is a disjoint union of affine derived schemes. Replacing X by U, we may suppose that X is a disjoint union of affine derived schemes. Replacing X by a sufficiently large finite union of components of X, we may suppose that $X = \mathrm{Spec}\, C$, where C is smooth over A. We must show that any A-algebra map $C \to B$ factors an étale neighborhood of B in A.

Locally C has the form of an étale algebra over $A[x_1, \ldots, x_m]$. Since $A \to B$ is surjective, the associated map $A[x_1, \ldots, x_m] \to B$ factors through A. Consequently, the map $C \to B$ factors through $C \otimes_{A[x_1, \ldots, x_m]} A$, which is an étale neighborhood of $\mathrm{Spec}\, B$ in $\mathrm{Spec}\, A$. □

We apply this to prove that if $A \to C$ and $B \to C$ are surjections in $\mathcal{SCR}$, then $\operatorname{Spec}(A \times_C B)$ may be interpreted as a pushout $\operatorname{Spec} A \coprod_{\operatorname{Spec} C} \operatorname{Spec} B$.

Theorem 5.6.4. *Suppose that $p : X \to Y$ is a relative stack. Let $A \to C$, $B \to C$ be surjective morphisms in $\mathcal{SCR}$ and $\eta \in Y(A \times_C B)$. Let $i : \operatorname{Spec} A \to \operatorname{Spec} A \times_C B$, $j : \operatorname{Spec} B \to \operatorname{Spec} A \times_C B$, and $k : \operatorname{Spec} C \to \operatorname{Spec} A \times_C B$ be the corresponding closed immersions of affine derived schemes.*

Let η_A, η_B, and η_C denote the corresponding elements of $Y(A)$, $Y(B)$, and $Y(C)$. Then the natural map $\phi : \mathcal{F}^p_\eta \to i_* \mathcal{F}^p_{\eta_A} \times_{k_* \mathcal{F}^p_{\eta_C}} j_* \mathcal{F}^p_{\eta_B}$ is an equivalence. In other words, the morphism p is cohesive.

Proof. Once again, we will deduce the theorem by applying Principle 5.3.5. Conditions (1) and (3) are obvious, so we just need to check condition (2). Let $U_0 \to X$ be a submersion, U_k denote the $(k+1)$-fold fiber power of U_0 over X, and let $q_k : U_k \to Y$ be the natural map. We suppose that the conclusion is known for each q_k. Then

$$\mathcal{F}^p_\eta = |\, \mathcal{F}^{q_\bullet}_\eta \,| = |i_* \mathcal{F}^{q_\bullet}_{\eta_A} \times_{k_* \mathcal{F}^{q_\bullet}_{\eta_C}} j_* \mathcal{F}^{q_\bullet}_{\eta_B} \,|.$$

Since $\mathcal{F}_n = i_* \mathcal{F}^{q_n}_{\eta_A} \times_{k_* \mathcal{F}^{q_n}_{\eta_C}} j_* \mathcal{F}^{q_n}_{\eta_B}$ is the $(n+1)$-fold fiber power of $\mathcal{F}_0$ over $i_* \mathcal{F}^p_{\eta_A} \times_{k_* \mathcal{F}^p_{\eta_C}} j_* \mathcal{F}^p_{\eta_B}$, it suffices to prove that

$$\mathcal{F}_0 \to i_* \mathcal{F}^p_{\eta_A} \times_{k_* \mathcal{F}^p_{\eta_C}} j_* \mathcal{F}^p_{\eta_B}$$

is surjective. To prove this, one begins with the surjection

$$k_* \mathcal{F}^{q_0}_{\eta_C} \to k_* \mathcal{F}^{q_n}_{\eta_C}$$

and applies Lemma 5.6.3 twice. ◻

Chapter 6

Formal Geometry

The purpose of this section is to sketch the development of formal geometry in the derived context, going so far as a version of Grothendieck's formal GAGA theorem. Strictly speaking, the results of this section are not needed in the proof of our representability theorem or to verify its hypotheses. Given a problem in "formal" derived algebraic geometry, we will generally be able to decouple the "formal" aspects from the "derived" aspects and treat them separately. However, one can just as easily treat them simultaneously using the ideas described in this section.

We begin in §6.1 with the requisite commutative algebra: $\mathfrak{p}$-adic topologies on simplicial commutative rings and the corresponding completion constructions. In §6.2 we consider a related, but slightly different, discussion of "pro-Artinian" completions. In particular, we prove a derived version of Schlessinger's criterion for the existence of formal versal deformation rings.

In §6.3 we give a proof of Grothendieck's formal GAGA theorem in the derived context. This is one instance in which the derived perspective offers a useful point of view: the fact that the "formal analytification" functor is fully faithful on coherent sheaves is deduced formally from the coherence theorem for proper direct images. The essential surjectivity does not seem to follow formally; however, it is easily reduced to the classical formal GAGA theorem.

In the last part of this section, §6.4, we prove that if X is a derived stack and A is complete, local, and Noetherian, then the space of A-valued points of X may be described as an inverse limit of spaces of A_α-valued points of X, where A_α ranges over a family of Artinian "quotients" of A. The significance of this result is that, according to Theorem 7.1.6, it is one of the defining characteristics of derived stacks.

6.1 Completions

Let $A \in \mathcal{SCR}$ and let $J \subseteq \pi_0 A$ be a finitely generated ideal. Choose a set of generators $\{x_1, \ldots, x_m\}$ for the ideal J. For each n, let $A_n = A \otimes_{\mathbf{Z}[y_1,\ldots,y_m]} \mathbf{Z}$, where each $y_i \mapsto x_i^{2^n} \in \pi_0 A$

and $y_i \mapsto 0 \in \mathbf{Z}$. We can naturally arrange an inverse system of A-algebras

$$\ldots \to A_2 \to A_1 \to A_0$$

We let $\hat{A}$ denote the corresponding object of $\text{Pro}(\mathcal{SCR})$. The following proposition shows that $\hat{A}$ depends only on the closed subset of the Zariski spectrum of $\pi_0 A$ determined by J, and not on J itself or the choice of generators:

Proposition 6.1.1. *Let $B \in \mathcal{SCR}$. Then the map $p : \text{Hom}_{\text{Pro}(\mathcal{SCR})}(\hat{A}, B) \to \text{Hom}_{\mathcal{SCR}}(A, B)$ is (-1)-truncated and expresses $\text{Hom}_{\text{Pro}(\mathcal{SCR})}(\hat{A}, B)$ as the union of those components of $\text{Hom}_{\mathcal{SCR}}(A, B)$ consisting of maps $f : A \to B$ for which $f(J^k) = 0 \in \pi_0 B$ for some $k \gg 0$.*

Proof. Let $f \in \text{Hom}_{\mathcal{SCR}}(A, B)$ be any morphism. If f is in the essential image of p, then clearly $f(J^k) = 0$ for $k \gg 0$. Suppose that the latter equation holds, and let Z_n denote the space of factorizations of f through A_n. We note that Z_n is nonempty if and only if $f(x_i^{2^n}) = 0$ for each $1 \leq i \leq m$, which is the case for $n \gg 0$. In this case, Z_n is a torsor for $(\Omega B)^m$, where ΩB denotes the loop space of the underlying space of A. In particular, $\pi_0 Z_n$ is a $(\pi_1 B)^m$ torsor and $\pi_i Z_n$ is naturally isomorphic to $(\pi_{i+1} B)^m$ for $i > 0$. By construction, the map $p : Z_n \to Z_{n+1}$ induces a constant map on π_0 and the zero map on π_{i+1} for $i \geq 0$. Consequently, the fiber of $\text{Hom}(\hat{A}, B) \to \text{Hom}(A, B)$ over f is the direct limit of the spaces Z_n as $n \to \infty$, which is contractible (by Whitehead's theorem, since the formation of homotopy groups is compatible with filtered colimits). □

If $\hat{A}$ is the J-adic completion of some $A \in \mathcal{SCR}$, then we may consider $\text{QC}_{\hat{A}}$, where we identify $\hat{A}$ with the corresponding functor $\mathcal{SCR} \to \mathcal{S}$. Equivalently, $\text{QC}_{\hat{A}}$ is the inverse limit of the ∞-categories $\text{QC}_{\text{Spec}\, A_n} = \mathcal{M}_{A_n}$. An object of $\text{QC}_{\hat{A}}$ may be represented by a family $\{M_n\}$ of $\{A_n\}$ modules, equipped with equivalences $M_n \simeq M_{n+1} \otimes_{A_{n+1}} A_n$. The utility of formal geometry is that the category $\text{QC}_{\hat{A}}$ is closely related to $\mathcal{M}_A$, particularly when A is J-adically complete. In order to explicate the relationship, we need to introduce some terminology.

Definition 6.1.2. Let $A \in \mathcal{SCR}$ and $J \subseteq \pi_0 A$ a finitely generated ideal. An A-module M is said to be *J-torsion* if for each $x \in \pi_k M$, $J^m x = 0 \in \pi_k M$ for $m \gg 0$. A module M is *J-acyclic* if $M \otimes_A A_1 = 0$, and *J-complete* if $\text{Hom}_A(N, M) = 0$ whenever N is J-acyclic.

Remark 6.1.3. The notions introduced above are borrowed from the theory of Bousfield localization in homotopy theory. They work especially well in this context because A_1 is perfect as an A-module.

Proposition 6.1.4. *Let $A \in \mathcal{SCR}$ and $J \subseteq \pi_0 A$ be a finitely generated ideal.*

1. *The class of J-torsion (J-acyclic, J-complete) modules depend only on the radical of J. Each constitutes a stable subcategory of the stable ∞-category of A-modules.*

2. *There exists an admissible t-structure (with trivial heart) on $\mathcal{M}_A$, with $(\mathcal{M}_A)_{\geq 0}$ and $(\mathcal{M}_A)_{\leq 0}$ given by the classes of J-acyclic and J-complete modules, respectively. In particular, for any $M \in \mathcal{M}_A$, there exists a morphism $M \to \hat{M}$ with J-complete target $\hat{M}$ and J-acyclic kernel.*

3. *The restriction map $r : \mathrm{QC}_A \to \mathrm{QC}_{\hat{A}}$ is zero on J-acyclic modules, and induces an equivalence of categories between the stable ∞-category of J-complete A-modules and the stable ∞-category $\mathrm{QC}_{\hat{A}}$.*

4. *The completion functor $M \mapsto \hat{M}$ induces an equivalence of categories between the stable ∞-category of J-torsion modules and the stable ∞-category of J-complete modules.*

Proof. Claim (1) is obvious for the class of J-torsion modules. For J-acyclic modules, we note that M is J-acyclic if and only if $rM = 0$, and $\hat{A}$ depends only on the radical of J. Finally, the class of J-acyclic modules determines the class of J-complete modules, which completes the proof of (1).

Let S denote the class of morphisms of A-modules which become equivalences after applying r. Since r commutes with all colimits, we can deduce that S is generated (as a saturated class of morphisms) by a set, so that (2) follows from the general theory of localizations: see for example [22].

Specifying a quasi-coherent complex M in $\mathrm{QC}_{\hat{A}}$ is equivalent to giving a system of A_n modules M_n together with equivalences $M_{n+1} \otimes_{A_{n+1}} A_n$. In particular, we may view $\{M_n\}$ as an inverse system of A-modules and form its inverse limit $\hat{M}$. The functor

$$M \mapsto \hat{M}$$

is right adjoint to r.

Since A_n is finitely presented as an A-module, tensoring with A_n commutes with limits. Thus $\hat{M} \otimes_A A_n$ is given by the limit of the system $\{M_{n+k} \otimes_A A_n\}$. A simple computation shows that the inverse system of homotopy groups $\{\pi_i M_{n+k} \otimes_A A_n\}$ is equivalent (as a pro-object) to $\pi_i M_n$. It follows that the natural map

$$r\hat{M} \to M$$

is an equivalence. Thus the completion functor $M \mapsto \hat{M}$ identifies $\mathrm{QC}_{\hat{A}}$ with a full subcategory of $\mathcal{M}_A$. To complete the proof of (3), it suffices to show that the essential image of the completion functor consists precisely of the J-complete modules. Since each M_n is J-complete as an A-module, the inverse limit $\hat{M}$ is necessarily J-complete. On the other hand, suppose that $\widetilde{M}$ is any J-complete A-module. Then the natural map $g : \widetilde{M} \to \widehat{r\widetilde{M}}$ induces an equivalence after applying r, so that the kernel of g is J-acyclic. Since both the source and target of g are J-complete, the kernel of g is J-complete. Consequently the kernel of g is zero and g is an equivalence, so that $\widetilde{M}$ lies in the essential image of the completion functor.

In view of (3), Claim (4) is equivalent to the assertion that r induces an equivalence of ∞-categories from the ∞-category of J-torsion A-modules to $\mathrm{QC}_{\hat{A}}$. We first show that r is fully faithful. One shows that the ∞-category of J-torsion modules is the smallest stable subcategory of $\mathcal{M}_A$ which contains every A_n and is stable under the formation of sums. Since r commutes with sums, it suffices to show that $\mathrm{Hom}_{\mathcal{M}_A}(A_n, M) = \mathrm{Hom}_{\mathcal{M}_{\hat{A}}}(rA_n, rM)$ for any $M \in \mathcal{M}_A$. The right hand side is given by the inverse limit of the spectra

$$\mathrm{Hom}_{\mathcal{M}_{A_k}}(A_n \otimes_A A_k, M \otimes_A A_k) = \mathrm{Hom}_{\mathcal{M}_A}(A_n, M \otimes_A A_k).$$

Since A_n is a perfect A-module, this is simply the J-completion of $M \otimes_A A_n^*$. However, since $M \otimes_A A_n^*$ is an A_n-module, it is already J-complete as an A-module, so that the mapping space in question is simply given by the 0th space of $M \otimes_A A_n^*$, which is $\mathrm{Hom}_A(A_n, M)$.

To complete the proof, it suffices to show that r is essentially surjective when restricted to J-torsion modules. If not, then there exists a nonzero complex $N \in \mathcal{M}_{\hat{A}}$ such that $\mathrm{Hom}_{\mathcal{M}_{\hat{A}}}(rM, N) = 0$ whenever M is J-torsion. Let $N = r\widetilde{N}$; then $\mathrm{Hom}_{\mathcal{M}_A}(M, \widetilde{N}) = 0$ whenever M is J-torsion. In particular, $\mathrm{Hom}_A(A_n^*, \widetilde{N}) = A_n \otimes_A \widetilde{N} = 0$ for each n, so that $N = r\widetilde{N} = 0$, a contradiction. □

Remark 6.1.5. The equivalence of ∞-categories provided by (4) of Proposition 6.1.4 is somewhat mysterious from the classical point of view. It is certainly not the case that every p-adically complete abelian group arises as the p-adic completion of a p-torsion group. For example, to obtain the group $\mathbf{Z}_p$ of p-adic integers, one must apply the left-derived functors of the p-adic completion to the group $\mathbf{Z}[\frac{1}{p}]/\mathbf{Z}$.

We next study the condition of J-completeness more carefully.

Proposition 6.1.6. *Let $A \in \mathcal{SCR}$, $M \in \mathcal{M}_A$, and let $J \subseteq \pi_0 A$ be a finitely generated ideal. The following conditions are equivalent:*

1. *The module M is J-complete.*
2. *For each $x \in J$, the module M is (x)-adically complete.*
3. *There exists a set of generators $\{x_1, \ldots, x_n\}$ for J such that M is (x_i)-adically complete for each i.*

Proof. If M is J-complete and N is (x)-acyclic for $x \in J$, then N is J-acyclic so that $\mathrm{Hom}(N, M) = 0$; thus M is (x)-complete. This proves $(1) \Rightarrow (2)$. The implication $(2) \Rightarrow (3)$ is obvious.

Let $\{x_1, \ldots, x_m\}$ be a system of generators for J. Suppose that M is (x_i)-adically complete for each i. We prove by induction on k that M is $J_k = (x_1, \ldots, x_k)$-adically complete. This is vacuous for $k = 0$. For $k > 0$, we let $\hat{A}_{J_k}$ denote the J_k-adic completion of A (as an object of $\mathrm{Pro}(\mathcal{SCR})$, and $\hat{A}_{k+1}$ the (x_{k+1})-adic completion of A; then $\hat{A}_{J_{k+1}} = \hat{A}_{J_k} \otimes_A \hat{A}_{k+1}$. We wish to prove that M is J_{k+1}-adically complete, so that M is equivalent to the inverse limit of the pro-system $M\otimes_A \hat{A}_{J_{k+1}} = M\otimes_A \hat{A}_{J_k}\otimes_A \hat{A}_{k+1}$. Since $\hat{A}_{k+1}$ may be taken to consist of perfect A-modules, tensoring with $\hat{A}_{k+1}$ commutes with limits, so that by the inductive hypothesis we deduce that the inverse limit of $M\otimes_A \hat{A}_{J_k}\otimes_A \hat{A}_{k+1}$ is equivalent to the inverse limit of $M\otimes_A \hat{A}_{k+1}$, which is equivalent to M since M is (x_{k+1})-adically complete. □

Proposition 6.1.7. *Let $A\in \mathcal{SCR}$, $J\subseteq \pi_0 A$ a finitely generated ideal, and M an A-module. The module M is J-complete if and only if each $\pi_i M$ is J-complete, when regarded as a discrete A-module.*

Proof. Using Proposition 6.1.6 we may reduce to the case where J is generated by a single element x. Suppose first that M is J-complete. Consider the long exact sequence associated to the triangle

$$M \xrightarrow{x^n} M \to M_n.$$

This gives rise to short exact sequences

$$0\to \pi_i M/(x^n\pi_n M)\to \pi_i M_n \to \ker(\pi_{i-1}M \xrightarrow{x^n} \pi_{i-1}M)\to 0.$$

Passing to the inverse limit, we deduce the existence of an exact sequence

$$0\to \lim{}^0\{\pi_i M/(x^n\pi_n M)\}\to \lim{}^0\{\pi_i M_n\}\to \lim{}^0\{\ker \pi_{i-1}M \xrightarrow{x^n} \pi_{i-1}M\}\to 0$$

and an isomorphism

$$\lim{}^1\{\pi_i M_n\}\simeq \lim{}^1\{\ker \pi_{i-1}M \xrightarrow{x^n} \pi_{i-1}M\}.$$

Let $\hat{M}$ denote the J-adic completion of M, so that there are exact sequences $0\to \lim^1\{\pi_{i+1}M_n\}\to \pi_i\hat{M}\to \lim^0\{\pi_i M_n\}\to 0$.

Since $M\simeq \hat{M}$, the natural map $\pi_i M\to \lim^0\{\pi_i M_n\}$ is surjective. Since this surjection factors through $\lim^0\{\pi_i M/(x^n\pi_n M)$, we deduce that $\lim^0\{\ker \pi_{i-1}M \xrightarrow{x^n} \pi_{i-1}M\} = 0$ so we have an exact sequence

$$0\to \lim{}^1\{\ker \pi_i M \xrightarrow{x^n} \pi_i M\}\to \pi_i M\to \lim{}^0\{\pi_i M/x^n\pi_i M\}\to 0.$$

The results of this calculation are unchanged if we replace M by $\pi_i M$, from which we may deduce that $\pi_i M$ is J-complete.

For the converse, suppose that each $\pi_i M$ is J-complete. Using the above calculations, one shows that the ith homotopy group of the J-adic completion of M depends only on $\pi_i M$ and $\pi_{i-1}M$. Thus, to show that $\pi_i M\simeq \pi_i\hat{M}$, we may suppose that M is i-truncated and $(i-2)$-connected. Then M is an extension of J-complete modules, hence J-complete. □

We note that there is a good theory of completions of Noetherian derived rings:

Proposition 6.1.8. *Let $A\in\mathcal{SCR}$ be Noetherian, and let $J\subseteq \pi_0 A$ be an ideal. The tautological morphism $\phi: A\to \lim \hat{A}$ is flat, and $\pi_0 \lim \hat{A}$ is the J-adic completion of $\pi_0 A$. In* *particular, ϕ is an equivalence if and only if $\pi_0 A$ is J-complete (in the usual sense), in which case we shall say that A is J-complete.*

Proof. Using induction on the number of generators of J, we may reduce to the case where J is generated by a single element $x \in \pi_0 A$. Let $i \geq 0$. There exists a map of inverse systems of (discrete) $\pi_0 A$-modules

$$\phi : \pi_i A/(x^{2^n} \pi_i A) \to \pi_i(A_n).$$

Since A is Noetherian, $\pi_i A$ is a finitely generated module over the Noetherian ring $\pi_0 A$. By the classical theory of completions of Noetherian rings, the inverse limit of this system is given by $\pi_i A \otimes_{\pi_0 A} \widehat{\pi_0 A}$, where $\widehat{\pi_0 A}$ is the J-adic completion of $\pi_0 A$. Moreover, $\widehat{\pi_0 A}$ is a flat $\pi_0 A$-module and all $\lim^1$-terms vanish. To complete the proof, it suffices to show that ϕ is an equivalence of pro-groups.

There exists a long exact sequence

$$\ldots \to \pi_i A \xrightarrow{x^{2^n}} \pi_i A \to \pi_i A_n \to \pi_{i-1} A \to \ldots.$$

From this, we see that ϕ is a monomorphism of proabelian groups with cokernel given by the pro-system $\{\ker \pi_{i-1} A \xrightarrow{x^{2^n}} \pi_{i-1} A\}$. Each morphism in this pro-system is zero, so the corresponding proabelian group is trivial. □

Remark 6.1.9. If $A \in \mathcal{SCR}$ is Noetherian and J-complete for some $J \subseteq \pi_0 A$, then any coherent A-module M is J-complete. By Proposition 6.1.7, it suffices to prove this in the case where M is discrete, and by Proposition 6.1.6 we may suppose that $J = (x)$. Then $\hat{M} = \lim\{\operatorname{coker} M \xrightarrow{x^n} M\}$. Consequently, we obtain the isomorphisms

$$\pi_1 \hat{M} \simeq \lim{}^0\{\ker \pi_0 M \xrightarrow{x^n} \pi_0 M\},$$

$$\pi_{-1} \hat{M} \simeq \lim{}^1\{\pi_0 M/(x^n \pi_0 M)\},$$

and an exact sequence

$$0 \to \lim{}^1\{\ker \pi_0 M \xrightarrow{x^n} \pi_0 M\} \to \pi_0 \hat{M} \to \lim{}^0\{\pi_0 M/(x^n)\pi_0 M\} \to 0$$

(and all other homotopy groups vanish). The first $\lim^1$-term vanishes because all the maps in the system are surjective. Since $\pi_0 M$ is a Noetherian $\pi_0 A$-module, the kernel of x^n on π_0 of M is constant for large M so that the inverse system of abelian groups $\{\ker \pi_0 M \xrightarrow{x^n} \pi_0 M\}$ is pro-trivial. We therefore deduce that $\hat{M}$ is discrete and $\pi_0 \hat{M}$ is the inverse limit of the abelian groups $\pi_0 M/(x^n \pi_0 M)$, which is the classical J-adic completion of M; the desired result now follows from classical commutative algebra.

Remark 6.1.10. One can model objects of $\mathcal{SCR}$ using topological commutative rings. In this case, the topology is merely a formal mechanism for discussing paths, homotopies of paths, and so forth. These topologies have nothing to do with the $\mathfrak{p}$-adic topologies on commutative rings (which are totally disconnected and have *no* nontrivial paths or homotopies), which play an important role in commutative algebra. The presence of "topology" in both aspects has the potential to be very confusing, so we shall try to avoid topological terminology in discussing pro-rings such as $\hat{A}$.

6.2 Pro-Artinian Completions

We shall say that an object $A \in \mathcal{SCR}$ is *Artinian* if $\pi_0 A$ is an Artinian ring, each $\pi_i A$ is a finite $\pi_0 A$ module, and $\pi_i A = 0$ for $i \gg 0$. Every Artinian object of $\mathcal{SCR}$ may be written (uniquely) as a finite product of local Artinian derived rings. In this section we will be exclusively concerned with the local case. If $A \in \mathcal{SCR}$ is local and Artinian, then its *residue field* is defined to be the residue field of $\pi_0 A$.

Let k be a field. We shall denote by $\mathcal{C}_k$ the ∞-category whose objects are local Artinian $A \in \mathcal{SCR}$, together with an identification of k with the residue field of A. We may view $\mathcal{C}_k$ as a full subcategory of $\mathcal{SCR}_{/k}$. A map in $\mathcal{C}_k$ is said to be *surjective* if it induces a surjection on π_0.

We now define the analogue of cohesive functors in the Artinian context. For simplicity, we shall restrict our attention to the case where $\mathcal{F}(k)$ is contractible. This involves no essential loss of generality, since a general functor $\mathcal{F}$ can be understood in terms of $\mathcal{F}(k)$ and the fiber of $\mathcal{F}$ over each point of $\mathcal{F}(k)$.

Definition 6.2.1. Let k be a field. A functor $\mathcal{F} : \mathcal{C}_k \to \mathcal{S}$ is *formally cohesive* if it satisfies the following conditions:

- The space $\mathcal{F}(k)$ is contractible.
- If $A \to C$ and $B \to C$ are surjective, then $\mathcal{F}(A \times_C B) \to \mathcal{F}(A) \times_{\mathcal{F}(C)} \mathcal{F}(B)$ is an equivalence.

In other words, a functor $\mathcal{F} : \mathcal{C}_k \to \mathcal{S}$ is formally cohesive if it commutes with most finite limits. We will see in a moment that this implies that $\mathcal{F}$ commutes with even more finite limits: in a sense, all finite limits which $\mathcal{C}_k$ ought to have. In order to prove this, we need to investigate the deformation theory of $\mathcal{F}$.

Lemma 6.2.2. *The functor $V \mapsto \mathcal{F}(k \oplus V)$, which is defined on perfect connective k-modules V, has a unique left exact extension to perfect k-modules.*

Proof. Define $\Omega(V)$ to be the nth loop space of $\mathcal{F}(k \oplus V[n])$ for $n \gg 0$. Since $\mathcal{F}$ is good, $\mathcal{F}(k \oplus V)$ is the loop space of $\mathcal{F}(k \oplus V[1])$ whenever V is connective. Thus $\Omega(V)$ is independent of the choice of n, so long as $V[n]$ is connective. If M is already connective, we may take $n = 0$, so we see that Ω extends the functor $V \mapsto \mathcal{F}(k \oplus V)$. It is clear by construction that Ω is the unique extension which commutes with the formation of loop spaces. To complete the proof, it suffices to show that Ω commutes with all finite limits. Clearly Ω preserves final objects, so we need only show that $\Omega(V' \times_V V'') \to \Omega(V') \times_{\Omega(V)} \Omega(V'')$. This follows from the equivalence

$$\mathcal{F}(k \oplus (V' \times_V V''[n])) \simeq \mathcal{F}(k \oplus V'[n]) \times_{\mathcal{F}(k \oplus V[n])} \mathcal{F}(k \oplus V''[n])$$

for n sufficiently large. □

Proposition 2.5.5 implies that the functor $V \mapsto \Omega(V)$ is given by the 0th space of $V \otimes T_{\mathcal{F}}$, for some uniquely determined k-module $T_{\mathcal{F}}$.We call $T_{\mathcal{F}}$ the *tangent complex* to $\mathcal{F}$. We note that $T_{\mathcal{F}}$ is covariantly functorial in $\mathcal{F}$. Its underlying spectrum may be described as follows: the nth space of $T_{\mathcal{F}}$ is given by $\mathcal{F}(k \oplus k[n])$. More concretely, the homotopy groups of $T_{\mathcal{F}}$ are given by the formula $\pi_i T_{\mathcal{F}} \simeq \pi_{i+j} \mathcal{F}(k \oplus k[j])$, valid for all $j \geq -i$.

Example 6.2.3. Let $\mathcal{F}_*(A) = \mathrm{Hom}_{\mathcal{C}_k}(k, A)$. Then $\mathcal{F}_*$ is a good functor (it is the initial object in the ∞-category of good functors $\mathcal{C}_k \to \mathcal{S}$). Its tangent complex $T_* = T_{\mathcal{F}_*}$ has homotopy groups which are zero in dimensions $\neq 0$. The k-vector space $\pi_0 T_*$ may be identified with the vector space of all derivations (over the prime field) from k into itself.

Remark 6.2.4. Let $\mathcal{D}$ denote the ∞-category of formally cohesive functors $\mathcal{C}_k \to \mathcal{S}$. An object of $\mathcal{D}$ may be thought of as a "formal neighborhood" of a k-valued point of some moduli space. Let $\mathcal{D}_* = \mathcal{D}_{/\mathcal{F}_*}$; we may think of objects of $\mathcal{D}_*$ as "formal neighborhoods" of k-valued points on moduli spaces defined over k.

Remark 6.2.5. When k is a field of characteristic zero, the $\mathcal{D}_*$ is the underlying ∞-category of the Quillen model category differential graded Lie algebras over k. If L is such a differential graded Lie algebra representing a functor $\mathcal{F} \in \mathcal{D}_*$, then the underlying differential graded vector space of L is a model for the kernel of $T_* \to T_{\mathcal{F}}$.

Lemma 6.2.6. *Let $f : A \to B$ be a surjective morphism in $\mathcal{C}_k$ which induces a surjection on π_0. Then there exists a factorization $A = B_n \to B_{n-1} \to \ldots \to B_0 = B$, where each $B_{i+1} \to B_i$ is a small extension with kernel $k[j]$ for some j.*

Proof. Let K be the cokernel of f, and let $n_B = \sum_i l(\pi_i K)$, where $l(\pi_i K)$ denotes the length of $\pi_i K$ as an Artinian module over the Artinian ring $\pi_0 A$. We prove Lemma 6.2.6 by induction on n_B. If $n_B = 0$, f is an equivalence and there is nothing to prove. If $n_B > 0$, then there exists some smallest value of i such that $\pi_i K \neq 0$. Since f is surjective, $i > 0$. Thus

$$\mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_i K) = \pi_i(K \otimes_A B) \simeq \pi_i L_{B/A},$$

and $\pi_j L_{B/A} = 0$ for $j < i$. Since $\pi_i K$ is nonzero, the group on the left is nonzero and therefore has a quotient which is length 1 (as a $\pi_0 A$-module). Consequently, we may construct a morphism of B-modules $L_{B/A} \to k[i]$; let B_1 denote the corresponding small extension of B over A. It is easy to see that $n_{B_1} = n_B - 1$, so that the inductive hypothesis implies the existence of a sequence of small extensions $A = B_n \to B_{n-1} \to \ldots \to B_1$. Appending the small extension $B_1 \to B$ we deduce the statement of the lemma. □

Lemma 6.2.7. *Let $\mathcal{F} : \mathcal{C}_k \to \mathcal{S}$ be a formally cohesive functor. If $f : A \to C$ and $g : B \to C$ are morphisms of $\mathcal{C}_k$, then $\mathcal{F}(A \times_C B) \to \mathcal{F}(A) \times_{\mathcal{F}(C)} \mathcal{F}(B)$ is an equivalence provided that either f or g is surjective.*

Proof. Without loss of generality we may suppose that f is surjective. Using Lemma 6.2.6, we may reduce to the case where f is a small extension by $k[j]$ for some $j \geq 0$. It suffices to treat the universal case where $f : k \to k \oplus k[j+1]$ is the zero section. In this case, g is also surjective, and the result follows. □

Proposition 6.2.8. *Let $\mathcal{F} \to \mathcal{F}'$ be a transformation of formally cohesive functors $\mathcal{C}_k \to \mathcal{S}$. The following conditions are equivalent:*

- *The map $T_{\mathcal{F}} \to T_{\mathcal{F}'}$ is 0-connected.*
- *For any surjection $A \to A'$ in $\mathcal{C}_k$, the induced map $\mathcal{F}(A) \to \mathcal{F}(A') \times_{\mathcal{F}'(A')} \mathcal{F}'(A)$ is surjective.*

Proof. Since any surjection is a composition of small extensions by shifts of k, (2) is equivalent to assertion that the corresponding statement holds for the universal such small extension, given by the zero section $k \to k \oplus k[j+1]$. In other words, (2) is equivalent to the assertion that the fiber of the map on zeroth spaces induced by $T_{\mathcal{F}}[j+1] \to T_{\mathcal{F}'}[j+1]$ is connected, for all $j \geq 0$. Clearly, this is equivalent to the assertion that $T_{\mathcal{F}} \to T_{\mathcal{F}'}$ is 0-connected. □

We shall say that a transformation $p : \mathcal{F} \to \mathcal{F}'$ of formally cohesive functors is *formally smooth* if it satisfies the equivalent conditions of Proposition 6.2.8.

Remark 6.2.9. The proof of Proposition 6.2.8 also shows that p is an equivalence if and only if it induces an equivalence $T_{\mathcal{F}} \to T_{\mathcal{F}'}$.

Let $A \in \mathcal{SCR}$. If A is local and Noetherian, then we will say that A is *complete* if it is $\mathfrak{m}$-complete, where $\mathfrak{m} \subseteq \pi_0 A$ is the maximal ideal.

Proposition 6.2.10. *For each $R \in \mathcal{SCR}_{/k}$, let $\mathcal{F}_R : \mathcal{C}_k \to \mathcal{S}$ be defined by*

$$\mathcal{F}_R(A) = \mathrm{Hom}_{\mathcal{SCR}_{/k}}(R, A).$$

1. *For any $R \in \mathcal{SCR}_{/k}$, the functor $\mathcal{F}_R$ is formally cohesive.*
2. *If $R \in \mathcal{SCR}_{/k}$, then $T_{\mathcal{F}_R}$ is the dual of the k-module $L_R \otimes_R k$.*
3. *If $R \in \mathcal{SCR}_{/k}$ is complete, local, and Noetherian with residue field k, then* $\mathrm{Hom}_{\mathcal{SCR}_{/k}}(R', R) = \mathrm{Hom}_{\mathcal{SC}_k}(\mathcal{F}_{R'}, \mathcal{F}_R)$ *for any $R' \in \mathcal{SCR}_{/k}$.*

Proof. Claims (1) and (2) are obvious. To prove (3), we let $\mathfrak{m}$ denote the maximal ideal of $\pi_0 R$, and let the $\mathfrak{m}$-adic completion of R be represented by the pro-system $\{R_i\}_{i\geq 0}$ of finitely presented R-algebras. Let $\hat{R}$ denote the pro-ring represented by the inverse system $\{\tau_{\leq j} R_i\}_{i,j\geq 0}$. We remark that $\hat{R}$ is *not* equal to the $\mathfrak{m}$-adic completion of R as pro-objects in general (although they have the same inverse limit). The pro-object $\hat{R}$ may be thought of as a pro-system in $\mathcal{C}_k$, and $\mathrm{Hom}_{\mathcal{SCR}_{/k}}(R, A) = \mathrm{Hom}(\hat{R}, A)$ for any $A \in \mathcal{C}_k$. Consequently, $\hat{R} \in \mathrm{Pro}\,\mathcal{C}_k$ corepresents the functor $\mathcal{F}_R$. Thus $\mathrm{Hom}_{\mathcal{G}\mathcal{C}_k}(\mathcal{F}_{R'}, \mathcal{F}_R) = \mathrm{Hom}_{\mathcal{SCR}_{/k}}(R', \lim \hat{R}) = \mathrm{Hom}_{\mathcal{SCR}_{/k}}(R', R)$, where the last equality follows from the fact that R is complete. □

Remark 6.2.11. Let $R \in \mathcal{SCR}$ be local and Noetherian with maximal ideal $\mathfrak{m} \subseteq \pi_0 R$, let $R' = \{R_n\}$ denote its $\mathfrak{m}$-completion (as an object in $\mathrm{Pro}(\mathcal{SCR})$), and $R'' = \{\tau_{\leq m} R_n\}$ its pro-Artinian completion (also as an object in $\mathrm{Pro}(\mathcal{SCR})$). There is a natural map $R' \to R''$ in $\mathrm{Pro}(\mathcal{SCR})$ which induces an equivalence after passing to the inverse limit: this follows from the fact that any object $A \in \mathcal{SCR}$ is given by the inverse limit of the tower $\{\tau_{\leq m} A\}$. The map $\phi : R' \to R''$ need not an equivalence in $\mathrm{Pro}(\mathcal{SCR})$. However, ϕ *is* an equivalence whenever R is k-truncated: in this case, the construction of $\{R_n\}$ given in §6.1 shows that we may take each R_n to be $(k+k')$-truncated, where k' is the number of generators of $\mathfrak{m}$. Consequently, the tower $\{\tau_{\leq m} R_n\}_{m\geq 0}$ is Pro-equivalent to R_n for each fixed n.

Our next goal is to prove Theorem 6.2.13, which characterizes the good functors having the form $\mathcal{F}_R$ for complete Noetherian $R \in \mathcal{SCR}$ with residue field k. Before we can prove this result, we need a simple lemma from classical commutative algebra. We include a proof for lack of a reference:

Lemma 6.2.12. *1. Let*

$$\ldots \to R_2 \to R_1 \to R_0$$

be an inverse system of (ordinary) local Artinian rings with the same residue field k. Denote the maximal ideal of R_i by $\mathfrak{m}_i$, and suppose that the induced maps $\mathfrak{m}_{i+1}/\mathfrak{m}_{i+1}^2 \to \mathfrak{m}_i/\mathfrak{m}_i^2$ on Zariski cotangent spaces are all injective. Then the inverse limit $R = \lim^0\{R_i\}$ is Noetherian.

2. Suppose that R is a complete local Noetherian ring and that

$$\ldots \to M_2 \to M_1 \to M_0$$

is an inverse system of finitely generated R-modules. Let $\mathfrak{m}$ be the maximal ideal of R, and suppose that each map $M_{i+1}/\mathfrak{m}M_{i+1} \to M_i/\mathfrak{m}M_i$ is injective. Then the inverse limit $M = \lim^0\{M_i\}$ is a finitely generated R-module.

Proof. We first prove (1). Passing to a subsequence if necessary, we may suppose that the sequence of Zariski cotangent spaces $\{\mathfrak{m}_i/\mathfrak{m}_i^2\}$ is constant, and consequently all of the maps $R_{i+1} \to R_i$ are surjective. Choose a map $A \to R$ where A is Noetherian and $A \to k$ is surjective. If k has characteristic zero, we may arrange that $A \simeq k$; in characteristic p we may take A to be the Witt vectors of k. Choose a finite collection of elements $\{y_i\} \subseteq R_0$

which generate $\mathfrak{m}_0/\mathfrak{m}_0^2$, and lift them to elements $\{\widetilde{y}_i\} \subseteq R$. There is a unique continuous ring homomorphism $A[[x_1, \ldots, x_n]] \to R$ which carries each x_i into $\widetilde{y}_i$. One checks that this homomorphism is surjective. Since $A[[x_1, \ldots, x_n]]$ is Noetherian, the quotient R is Noetherian.

To prove (2), we again begin by passing to a subsequence so that the inverse system of vector spaces $\{M_i/\mathfrak{m}M_i\}$ is constant. By Nakayama's lemma, we see that each map $M_{i+1} \to M_i$ is surjective. Consequently, we may choose a finite collection $\{x_1, \ldots, x_n\} \subseteq M$ whose images form a basis for $M_0/\mathfrak{m}M_0$. Using the fact that R is complete, one shows that the sequence $\{x_1, \ldots, x_n\}$ generates M. □

To simplify the statement and proof of the next theorem, we introduce a notation for *relative tangent complexes*. Given a transformation $\mathcal{F}' \to \mathcal{F}$ of good functors, we define $T_{\mathcal{F}'/\mathcal{F}}$ to be the kernel of $T_{\mathcal{F}'} \to T_{\mathcal{F}}$. If $\mathcal{F}' = \mathcal{F}_{R'}$ for some $R' \in \mathcal{SCR}_{/k}$, then we will abbreviate by simply writing $T_{R'/\mathcal{F}}$; similarly if $\mathcal{F} = \mathcal{F}_R$ then we shall denote the tangent complex simply by $T_{R'/R}$. We note that $T_{R'/R} \simeq \mathrm{Hom}_{\mathcal{M}_{R'}}(L_{R'/R}, k)$.

We now come to the main result of this section: the derived version of Schlessinger's criterion. This gives precise conditions under which a functor $\mathcal{C}_k \to \mathcal{S}$ is representable by a complete local Noetherian $R \in \mathcal{SCR}$ having residue field k. This is the "infinitesimal" ingredient in the proof of our main result, the derived version of Artin's representability theorem. Strictly speaking, we do not need Theorem 6.2.13 in the proof of Theorem 7.1.6: the classical version of Schlessinger's criterion will be sufficient for our purposes. However, the derived formulation of this result is interesting enough in its own right:

Theorem 6.2.13 (Derived Schlessinger Criterion). *Let $\mathcal{F}$ be a good functor $\mathcal{C}_k \to \mathcal{S}$. The following conditions are equivalent:*

1. *There exists a complete local Noetherian $R \in \mathcal{SCR}_{/k}$, having residue field k, and a smooth transformation $\mathcal{F}_R \to \mathcal{F}$.*

2. *The k-vector spaces $\pi_i T_{k/\mathcal{F}}$ are finite dimensional for $i < 0$.*

Proof. If $\mathcal{F} \to \mathcal{F}'$ is smooth and $\mathcal{F}$ satisfies (2), then so does $\mathcal{F}'$. It is clear that (2) is satisfied when $\mathcal{F} = \mathcal{F}_R$, where R is local, Noetherian, and has residue field k. Thus (1) implies (2).

The hard part is to show that (2) implies (1). We will construct a sequence of complete local Noetherian objects $R^i \in \mathcal{SCR}$, such that $R^i = \tau_{\leq i} R^{i+1}$, and a compatible family of transformations $\phi^i : \mathcal{F}_{R^i} \to \mathcal{F}$ such that $\pi_n T_{R^i/\mathcal{F}} = 0$ for $0 < n \leq (-i-1)$. Assuming that this is possible, we set $R = \lim\{R^i\}$. Then $\mathcal{F}_R = \mathrm{colim}\{\mathcal{F}_{R^i}\}$, so that the compatible family of transformations ϕ^i gives rise to a smooth transformation $\phi : R \to \mathcal{F}$ as desired.

The construction proceeds by induction on i. Let us begin with the case $i = 0$. We give an argument which is essentially identical to the proof of the main theorem of [30]. We construct the ordinary Noetherian ring R^0 as the inverse limit of a sequence of local Artinian algebras R^0_j equipped with maps $\phi^0_j : \mathcal{F}_{R^0_j} \to \mathcal{F}$, which we may identify with elements of $\mathcal{F}(R^0_j)$. We begin by setting $R^0_0 = k$, and $\phi^0_0 = * \in \mathcal{F}(k)$.

Assuming that R_j^0 and ϕ_j^0 have already been constructed, let V denote the k-vector space which is given by $\pi_{-1}T_{R_j^0/\mathcal{F}}$. Choose a point η in the 0th space of $T_{R_j^0/\mathcal{F}} \otimes_k V^*[1]$ in the connected component classifying the canonical element of $V \otimes_k V^*$ (if the homotopy groups of $T_{k/\mathcal{F}}$ vanish in positive degrees, then η is essentially unique so that the construction is functorial; otherwise we must make an arbitrary choice). Then η classifies a morphism $R_j^0 \to R_j^0 \oplus V^*[1]$ over $\mathcal{F}$. Let R_{j+1}^0 denote the corresponding square-zero extension of R_j^0 by V^*. Since $\mathcal{F}$ is a good functor, we get a natural map $\phi_{j+1}^0 : \mathcal{F}_{R_{j+1}^0} \to \mathcal{F}$ which extends ϕ_j^0.

Next we claim that the projection $R_{j+1}^0 \to R_j^0$ induces an surjection on Zariski tangent spaces for $j \geq 1$. To see this, consider the exact sequence

$$\pi_{-1}T_{k/R_j^0} \xrightarrow{f} \pi_{-1}T_{k/R_{j+1}^0} \to \pi_{-1}T_{R_j^0/R_{j+1}^0} \xrightarrow{g} \pi_{-2}T_{k/R_j^0}$$

$$\pi_{-1}T_{k/R_j^0} \xrightarrow{f'} \pi_{-1}T_{k/\mathcal{F}} \to \pi_{-1}T_{R_j^0/\mathcal{F}} \xrightarrow{g'} \pi_{-2}T_{k/R_j^0}.$$

The surjectivity of f is equivalent to the injectivity of g. Since we have $\pi_{-1}T_{R_j^0/R_{j+1}^0} \simeq \pi_{-1}T_{R_j^0/\mathcal{F}}$ by construction, it suffices to show that g' is injective, which is equivalent to the surjectivity of f'. In order to show that f' is surjective, it suffices to show that the composition

$$\pi_{-1}T_{k/R_1^0} \to \pi_{-1}T_{k/R_j^0} \xrightarrow{f'} \pi_{-1}T_{k/\mathcal{F}}$$

is a surjection. But this composite map is an isomorphism by construction.

Now we may apply Lemma 6.2.12 to conclude that the inverse limit R^0 of the sequence

$$\ldots \to R_2^0 \to R_1^0 \to R_0^0$$

is Noetherian. Moreover, the functor $\mathcal{F}_{R^0}$ is the filtered colimit of the functors $\mathcal{F}_{R_j^0}$, so that we get a natural map $\phi^0 : \mathcal{F}_{R^0} \to \mathcal{F}$. Moreover, the relative tangent complex $T_{R^0/\mathcal{F}}$ is the filtered colimit of the relative tangent complexes $T_{R_j^0/\mathcal{F}}$. By construction, the natural map $\pi_{-1}T_{R_j^0/\mathcal{F}} \to \pi_{-1}T_{R_{j+1}^0/\mathcal{F}}$ is zero for all j, so that the filtered colimit in question is trivial and the construction of R^0 is complete.

The construction of R^{i+1} for $i \geq 0$ is similar. Namely, we first construct a sequence of R_j^{i+1} together with maps $\phi_j^{i+1} : \mathcal{F}_{R_j^{i+1}} \to \mathcal{F}$. We begin by setting $R_0^{i+1} = R^i$ and $\phi_0^{i+1} = \phi^i$. For each $j \geq 0$, we let V denote the k-vector space $\pi_{-i-2}T_{R_j^{i+1}/\mathcal{F}}$, and we choose a point η in the 0th space of $T_{R_j^i/\mathcal{F}} \otimes_k V^*[i+2]$ lying in the connected component of $\mathrm{id}_V \in V \otimes_k V^*$. As above, the element η classifies a morphism $R_j^{i+1} \to R_j^{i+1} \oplus V^*[i+2]$ over $\mathcal{F}$, and we take R_{j+1}^{i+1} to be the corresponding square-zero extension of R_j^{i+1} by $V^*[i+1]$. By construction, this comes equipped with a canonical lifting ϕ_{j+1}^{i+1} of ϕ_j^{i+1}.

Let M_j denote the R^0-module $\pi_{i+1}R_j^{i+1}$. By construction, $M_0 = 0$, and M_{j+1} is an extension of M_j by a finite dimensional k-vector space. One next shows that $M_j/\mathfrak{m}M_j \to M_{j-1}/\mathfrak{m}M_{j-1}$ is injective for $j \geq 2$, where $\mathfrak{m}$ denotes the maximal ideal of R^0. Lemma 6.2.12 now applies to show that $M = \lim\{M_j\}$ is a finitely generated R^0 module (and discrete; the relevant $\lim^1$-term vanishes since each map $M_{j+1} \to M_j$ is surjective). Now set $R^{i+1} = \lim\{R_j^{i+1}\}$. It is clear that R^{i+1} is $(i+1)$-truncated, $\tau_{\leq i} R^{i+1} \simeq R^i$, and $\pi_{i+1} R^{i+1} = M$ so that R^{i+1} is Noetherian. Moreover, $\mathcal{F}_{R^{i+1}}$ is the filtered colimit of the functors $\mathcal{F}_{R_j^{i+1}}$, so the compatible family ϕ_j^{i+1} gives rise to map $\phi^{i+1} : \mathcal{F}_{R^{i+1}} \to \mathcal{F}$ which lifts ϕ^i. Moreover, $T_{R^{i+1}/\mathcal{F}}$ is the filtered colimit of the complexes $T_{R_j^{i+1}/\mathcal{F}}$. We have $\pi_n T_{R_j^{i+1}/\mathcal{F}} = 0$ for $0 < n \leq (-1-i)$, and the transition maps for the direct system $\{\pi_{-2-i} T_{R_j^{i+1}/\mathcal{F}}\}$ are all equal to zero by construction, so that the direct limit vanishes. This completes the proof of the theorem. □

In the case where $\pi_i T_{k/\mathcal{F}} = 0$ for $i > 0$, we can be more precise:

Corollary 6.2.14. *Let $\mathcal{F}$ be a good functor $\mathcal{C}_k \to \mathcal{S}$. The following conditions are equivalent:*

1. *There exists a complete local Noetherian $R \in \mathcal{SCR}_{/k}$ with residue field k and an equivalence $\mathcal{F}_R \simeq \mathcal{F}$.*

2. *The vector spaces $T_{k/\mathcal{F}}$ are finite dimensional for $i < 0$ and vanish for $i \geq 0$.*

Proof. It is clear that (1) implies (2). For the reverse implication, we apply Theorem 6.2.13 to deduce the existence of a smooth morphism of good functors $\mathcal{F}_R \to \mathcal{F}$, where $R \in \mathcal{SCR}_{/k}$ is complete, local, and Noetherian with residue field k. Let K denote the kernel of the map of k-vector spaces $\pi_0 T_{\mathcal{F}_R} \to \pi_0 T_{\mathcal{F}}$. Since $\pi_0 T_*$ injects into $\pi_0 T_{\mathcal{F}}$, K does not intersect $\pi_0 T_* \subseteq \pi_0 T_{\mathcal{F}_R}$. Consequently, K is a finite dimensional vector space, and we have a surjective map $\pi_0(L_{R/\mathbf{Z}} \otimes_R k) \to K^* \oplus \pi_0(L_{k/\mathbf{Z}})$. It follows that there is a surjection $\pi_1 L_{k/R} \to K^*$. On the other hand, $\pi_1 L_{k/R} \simeq \mathfrak{m}/\mathfrak{m}^2$ where $\mathfrak{m}$ denotes the maximal ideal of $\pi_0 R$. Consequently, we may choose a finite sequence $\{x_1, \ldots, x_m\} \subseteq \mathfrak{m}$ which maps to a basis for K^*. Let $R' \in \mathcal{SCR}_{/k}$ be the R-algebra $R \otimes_{\mathbf{Z}[x_1,\ldots,x_m]} \mathbf{Z}$, obtained by killing the x_i. Then the composite $\mathcal{F}_{R'} \to \mathcal{F}_R \to \mathcal{F}$ induces an equivalence on tangent complexes and is therefore an equivalence. □

Remark 6.2.15. Alternatively, one can observe that Corollary 6.2.14 follows from the proof of Theorem 6.2.13, rather than its conclusion.

6.3 Formal GAGA

Let $A \in \mathcal{SCR}$ be Noetherian and let $\hat{A}$ denote its J-adic completion, for some ideal $J \subseteq \pi_0 A$. We will let $\operatorname{Spf} \hat{A}$ denote the functor $\mathcal{SCR} \to \mathcal{S}$ represented by the pro-object $\hat{A}$. We could instead define $\operatorname{Spf} \hat{A}$ as an ∞-topos equipped with a certain $\operatorname{Pro}(\mathcal{SCR})$-valued sheaf as part of a more general theory of *formal derived schemes*, but this seems to be more trouble than it is worth; we will only need to discuss formal derived schemes which are "of finite presentation" over affine models having the form $\operatorname{Spf} \hat{A}$, and these can be described in terms of relative derived schemes $X \to \operatorname{Spf} A$.

The point of this section is to deduce a version of Grothendieck's formal GAGA theorem. The classical version of Grothendieck's theorem asserts that if A is a Noetherian ring which is complete with respect to an ideal J, and X is a scheme proper over $\operatorname{Spec} A$, then the category of coherent sheaves on X is equivalent to the category of coherent sheaves on the formal scheme $\hat{X}$ which is obtained by formal completion along the ideal J. We will prove a similar result in the derived setting. Let us begin by considering an arbitrary Noetherian $A \in \mathcal{SCR}$, and an arbitrary derived scheme $X = (\mathcal{X}, \mathcal{O})$ over $\operatorname{Spec} A$. For any $U \in \mathcal{X}$, we let X_U denote the derived scheme $(\mathcal{X}_{/U}, \mathcal{O}|U)$. If $J \subseteq \pi_0 A$ is an ideal, we let $\hat{X}$ denote the fiber product $X \times_{\operatorname{Spec} A} \operatorname{Spf} \hat{A}$, and $\hat{X}_U = X_U \times_{\operatorname{Spec} A} \operatorname{Spf} \hat{A}$. We note that if $X_U \simeq \operatorname{Spec} B$, then $\hat{X}_U \simeq \operatorname{Spf} \hat{B}$, where the completion is taken with respect to the $J(\pi_0 B)$-adic topology. In particular, we note that quasi-coherent complexes on $\hat{X}_U$ may be identified with J-complete B-modules. Passing from local to global, we may identify quasi-coherent complexes M on $\hat{X}$ with functors associating a J-complete B-module $M(\eta)$ to each étale map $\eta : \operatorname{Spec} B \to U$. The functoriality takes the form of functorial equivalences $M(\eta') = \widehat{M(\eta) \otimes_B} C$ whenever C is an étale B-algebra (and η' the induced map $\operatorname{Spec} C \to X$). In particular, we may endow $\mathrm{QC}_{\hat{X}}$ with a t-structure, given by patching together the natural t-structures on $\mathrm{QC}_{\operatorname{Spf} \hat{B}} \subseteq \mathcal{M}_B$ for all étale maps $\operatorname{Spec} B \to X$. We may also speak of *coherent* objects of $\mathrm{QC}_{\hat{X}}$, which are given by those complexes which are locally given by coherent modules over the completion $\lim \hat{B}$.

Theorem 6.3.1. *Let $A \in \mathcal{SCR}$ be Noetherian and J-complete for some ideal $J \subseteq \pi_0 A$. Let X be a derived scheme which is proper over $\operatorname{Spec} A$, and let $\hat{X} = X \times_{\operatorname{Spec} A} \operatorname{Spf} \hat{A}$. Then the restriction induces an equivalence between the ∞-categories of coherent complexes on X and $\hat{X}$.*

Proof. We first show that the restriction functor is fully faithful. Let $M, N \in \mathrm{QC}_X$ be coherent; we wish to show that $\operatorname{Hom}_{\mathrm{QC}_X}(M, N) \to \operatorname{Hom}_{\mathrm{QC}_{\hat{X}}}(M|\hat{X}, N|\hat{X})$ is an equivalence. Both sides are compatible with colimits in M, so we may reduce to the case where M is almost perfect.

Now $N \simeq \lim\{\tau_{\leq n} N\}$ and $N|\hat{X} \simeq \lim\{(\tau_{\leq n} N)|\hat{X}\}$, where the second equivalence follows from the fact that $\operatorname{Spf} A$ may be represented by an inverse system of A-algebras with uniformly bounded Tor-amplitude over A. Using these equivalences, we may reduce to the case where N is truncated. Now we may form a mapping complex $K = \operatorname{Hom}(M, N)$, which is compatible with base change; $\operatorname{Hom}_{\mathrm{QC}_X}(M, N)$ is given by the global sections of K, and $\operatorname{Hom}_{\mathrm{QC}_{\hat{X}}}(M|\hat{X}, N|\hat{X})$ is given by the global sections of $K|\hat{X}$.

Let $\hat{A}$ be the inverse limit of almost perfect A-algebras $\{A_n\}$, let $X_n = X \times_{\operatorname{Spec} A} \operatorname{Spec} A_n$, and let $p_n : X_n \to \operatorname{Spec} A_n$ be the projection. We wish to show that $p_* K \simeq \lim\{(p_n)_* K|X_n\}$. Since K is truncated, the push-pull formula gives $(p_n)_*(K|X_n) \simeq (p_* K) \otimes_A A_n$; thus it suffices to show that $p_* K$ is J-complete. This follows from the coherence of $p_* K$, since A is J-complete.

For the essential surjectivity, let us consider any coherent $M \in \mathrm{QC}_{\hat{X}}$. We wish to show that M is formal completion of some coherent $M_0 \in \mathrm{QC}_X$. By passing to direct limits, we can reduce to the case where M is almost perfect. Passing to inverse limits, we may suppose that M is truncated. Working by induction on the number of nonzero homotopy sheaves, we may reduce to the case where M is discrete. We are thereby reduced to the case of an coherent sheaf on a proper formal Deligne-Mumford stack, which is handled in [20]. □

6.4 A Comparison Theorem

One of the usual applications of Grothendieck's formal GAGA theorem is the following: if A is a J-complete Noetherian ring, Y is a scheme which is proper over $\operatorname{Spec} A$, and X is a scheme which is separated (and finite type) over $\operatorname{Spec} A$, then $\operatorname{Hom}_{\operatorname{Spec} A}(Y, X) \simeq \operatorname{Hom}_{\operatorname{Spf} A}(\hat{Y}, \hat{X})$, where $\hat{X}$ and $\hat{Y}$ denote the formal completions of X and Y along J. We will prove a derived version of this result in [23], under the assumption that X is a *geometric stack* (a class of objects which includes all separated derived algebraic spaces).

The purpose of this section is to prove a slightly different (and generally less useful) variant on this result, in which we allow X to be arbitrary but require that A be local, $J = \mathfrak{m}$ the maximal ideal of A, and $Y = \operatorname{Spec} A$. In this situation, we have:

Theorem 6.4.1. *The natural map $X(A) \to \lim\{X(A_n)\}$ is an equivalence.*

In order to prove the Theorem, we must first give an equivalent formulation in slightly fancier language. Let $\mathcal{Y}$ denote the underlying ∞-topos of $\operatorname{Spec} A$, and let $\mathcal{Y}_0$ denote the underlying ∞-topos of $\operatorname{Spec} k$, where k is the residue field of A. We note that $\mathcal{Y}_0$ may also be identified (canonically) with the underlying ∞-topos of $\operatorname{Spec} A_n$ for each n (here, and in what follows, we let A_n be defined as at the beginning of §6.1). Let $\pi : \mathcal{Y}_0 \to \mathcal{Y}$ denote the geometric morphism induces by the quotient map $A \to k$.

The restriction of X to the category of étale A_n-algebras gives an object $\mathcal{F}_n^X \in \mathcal{Y}_0$; similarly, the restriction of X to étale A-algebras gives an object $\mathcal{F}^X \in \mathcal{X}$. There is a natural map $\phi : \pi^* \mathcal{F}^X \to \lim\{\mathcal{F}_n^X\}$. We will actually show:

Theorem 6.4.2. *The map ϕ is an equivalence of objects of $\mathcal{Y}_0$.*

In order to relate Theorem 6.4.1 from Theorem 6.4.2, we need the following lemma whose proof is left to the reader.

Lemma 6.4.3. *Let k' be an étale k-algebra, and let A'_n denote the (essentially unique) étale A_n-algebra with $A'_n \otimes_{A_n} k \simeq k'$. Let A' denote the inverse limit of the system $\{A'_n\}$. Then A' is an étale A-algebra, and for any $\mathcal{F} \in \mathcal{Y}$ we have a natural equivalence $\mathcal{F}(A') \simeq (\pi^* \mathcal{F})(k')$.*

Using Lemma 6.4.3, it is easy to see that Theorem 6.4.2 holds for A if and only if Theorem 6.4.1 holds for every finite étale A-algebra (which are precisely the algebras of the form A' as in the statement of Lemma 6.4.3).

We can now give a proof of Theorems 6.4.2 and 6.4.1. For any derived stack X, let P be the assertion that Theorem 6.4.2 holds for *any* complete local Noetherian $A \in \mathcal{SCR}$. We wills how that all derived stacks have the property P by applying Principle 5.3.5. We must check that conditions (1) through (3) of Principle 5.3.5 are satisfied:

1. We must show that Theorem 6.4.2 holds when $X = \operatorname{Spec} B$ is affine. This is obvious from the formulation given in Theorem 6.4.1, since the completeness assumption guarantees that $A = \lim\{A_n\}$.

2. Suppose that we are given a submersion $U_0 \to X$ and that the theorem is known for each U_k, where U_k denotes the $(k+1)$-fold fiber power of U_0 over X. We note that $\mathcal{F}_\infty^{U_k}$ is the $(k+1)$st fiber power of $\mathcal{F}_\infty^{U_0}$ over $\mathcal{F}_\infty^X$, and the assumption tells us that this is naturally equivalent to $\pi^* \mathcal{F}^{U_k}$ which is the $(k+1)$st fiber power of $\pi^* \mathcal{F}^{U_0}$ over $\pi^* \mathcal{F}^X$. This proves that $\pi^* \mathcal{F}^X \to \mathcal{F}_\infty^X$ is (-1)-truncated. To complete the proof, it suffices to show that $\mathcal{F}_\infty^{U_0} \to \mathcal{F}_\infty^X$ is surjective. Suppose that a section $\mathcal{F}_\infty^X$ is given over some étale k-algebra k'. Passing to some cover of k' if necessary, we may suppose that the corresponding map $\eta_0 : \operatorname{Spec} A'_0 \to X$ factors through some map $\widetilde{\eta}_0 : \operatorname{Spec} A'_0 \to U_0$. To complete the proof, it suffices to show that each extension of $\eta_n : \operatorname{Spec} A'_n \to X$ to $\eta_{n+1} : \operatorname{Spec} A'_n \to X$ can be covered by an extension of $\eta'_n : \operatorname{Spec} A'_n \to U_0$ to $\eta'_{n+1} : \operatorname{Spec} A'_n \to U_0$. This follows immediately from the fact that U_0 is smooth over X.

3. Assuming that Theorem 6.4.1 holds for a sequence of open subfunctors $X_{(0)} \subseteq \ldots$, we must show that it holds for their union. It suffices to show that $X(A) = \operatorname{colim}\{X_{(n)}(A)\}$ and that $\lim\{X(A_m)\} = \operatorname{colim} \lim\{X_{(n)}(A_m)\}$. The first claim is obvious (since $\operatorname{Spec} A$ is compact), and the second follows from the compactness of $\operatorname{Spec} A_n$ together with the fact that the topology of $\operatorname{Spec} A_n$ is independent of n.

This completes the proof of Theorem 6.4.1.

Chapter 7

The Representability Theorem

7.1 Review of Artin's Theorem

Let $\mathcal{F}$ be a set-valued (covariant) functor defined on the category of commutative R-algebras, where R is a fixed commutative ring. A basic question is whether or not $\mathcal{F}$ is representable by some geometric object $X \to \operatorname{Spec} R$, in the sense that $\mathcal{F}(B) = \operatorname{Hom}_{\operatorname{Spec} R}(\operatorname{Spec} B, X)$. Of course, the answer to this question depends on what class of geometric objects we allow ourselves to consider. Artin's representability theorem (see [2]) asserts that we can take X to be an algebraic space, provided that the ring R is sufficiently nice and the functor $\mathcal{F}$ satisfies certain criteria. More generally, Artin allows $\mathcal{F}$ to be groupoid-valued, which case the class of representing geometric objects must be enlarged to include algebraic stacks (see [2]).

This result is of both philosophical and practical interest. Since Artin's criteria are obviously necessary for the existence of a reasonable geometric representation of $\mathcal{F}$, the sufficiency gives evidence that the class of algebraic spaces (or, more generally, algebraic stacks) is a good class of objects to consider. On the other hand, if we are given a functor $\mathcal{F}$, it is usually reasonably easy to check whether or not Artin's criteria are satisfied. Consequently Artin's theorem can be used to build a great number of moduli spaces.

Before proceeding into greater detail, we recall the statement of Artin's theorem:

Theorem 7.1.1 (Artin). *Let R be an excellent Noetherian ring, and let $\mathcal{F}$ be functor from (ordinary) R-algebras to groupoids. Then $\mathcal{F}$ is representable by an Artin stack which is locally of finite presentation over R if and only if the following conditions are satisfied:*

1. *The functor $\mathcal{F}$ commutes with filtered colimits.*

2. *The functor $\mathcal{F}$ is a sheaf (of groupoids) for the étale topology.*

3. *If B is a complete local Noetherian R-algebra with maximal ideal $\mathfrak{m}$, then the natural map*

$$\mathcal{F}(B) \to \lim\{\mathcal{F}(B/\mathfrak{m}^n)\}$$

   is an equivalence.

4. *The functor $\mathcal{F}$ admits an obstruction theory and a deformation theory, and satisfies Schlessinger's criteria for formal representability.*

5. *The diagonal map $\mathcal{F} \to \mathcal{F} \times_{\operatorname{Spec} R} \mathcal{F}$ is representable by an algebraic space.*

Remark 7.1.2. The original formulation of Artin's theorem (see [2]) had a more restrictive hypothesis than excellence on the ring R. For a careful discussion of the removal of this hypothesis, we refer the reader to [8].

Remark 7.1.3. In the original formulation of the representability theorem, condition (3) was replaced by the apparently weaker assumption that the natural map have dense image (with respect to the inverse limit topology on the target). The extra generality tends not to be so useful in practice, since the stronger version of (3) is usually just as easy to verify. More importantly, the density assumption is not so natural once we begin to consider moduli functors which are valued in ∞-groupoids.

We refer the reader to [2] for the precise meaning of assumption (4). We merely remark that the obstruction and deformation theory are additional data which are related to, but not (uniquely) determined by, the functor $\mathcal{F}$. The meaning of this additional data is much better understood from the derived point of view: it has to do with extending the functor $\mathcal{F}$ to a small class of nondiscrete R-algebras. In the derived setting, we will suppose that we are given a functor $\mathcal{F}$ which is defined on *all* of $\mathcal{SCR}_{R/}$. In this case, the analogue of condition (4) is that the functor $\mathcal{F}$ be infinitesimally cohesive and possess a cotangent complex over R. This assumption is much more conceptually satisfying, since the cotangent complex of $\mathcal{F}$ *is* uniquely determined by $\mathcal{F}$.

The main theorem of this paper is the derived analogue of Theorem 7.1.1. Our result will give necessary and sufficient conditions for the representability of a $\mathcal{S}$-valued functor on $\mathcal{SCR}_{R/}$, where $R \in \mathcal{SCR}$. Our proof will require some technical hypotheses on R.

Definition 7.1.4. An object $R \in \mathcal{SCR}$ is a *derived G-ring* if the following conditions are satisfied:

- R is Noetherian.
- For each prime ideal $\mathfrak{p} \subseteq \pi_0 R$, the $\mathfrak{p}R_{\mathfrak{p}}$-adic completion of $R_{\mathfrak{p}}$ is a geometrically regular R-algebra.

In other words, a derived G-ring is a Noetherian object R of $\mathcal{SCR}$ such that $\pi_0 R$ is a G-ring in the usual sense (see [26]). Since the class of discrete G-rings is stable under the formation of finitely presented ring extensions (by a theorem of Grothendieck: see [26]), we deduce that the class of derived G-rings is stable under passage to almost finitely presented extensions.

Remark 7.1.5. We could similarly define an object $R \in \mathcal{SCR}$ to be *excellent* if it is Noetherian, and $\pi_0 R$ is an excellent ring in the usual sense (see [26]). Excellence is a more common (and stronger) hypothesis than the condition of being a G-ring; however, we shall not need this stronger condition.

We are now in the position to state our theorem:

Theorem 7.1.6. *Let R be a derived G-ring. Let $\mathcal{F} : \mathcal{SCR}_{R/} \to \mathcal{S}$ a functor. Then $\mathcal{F}$ is representable by a derived n-stack which is almost of finite presentation over R if and only if the following conditions are satisfied:*

1. *The functor $\mathcal{F}$ commutes with filtered colimits when restricted to k-truncated objects of $\mathcal{SCR}_{R/}$, for each $k \geq 0$.*
2. *The functor $\mathcal{F}$ is a sheaf for the étale topology.*
3. *Let B be a complete, discrete, local, Noetherian R-algebra, $\mathfrak{m} \subseteq B$ the maximal ideal. Then the natural map $\mathcal{F}(B) \to \lim\{\mathcal{F}(B/\mathfrak{m}^n)\}$ is an equivalence.*
4. *The functor $\mathcal{F}$ has a cotangent complex.*
5. *The functor $\mathcal{F}$ is infinitesimally cohesive.*
6. *The functor $\mathcal{F}$ is nilcomplete.*
7. *For any discrete commutative ring R, the space $\mathcal{F}(R)$ is n-truncated.*

The proof of the "if" direction will be given in §7.3. The remainder of this section is devoted to a discussion of conditions (1) through (7) of Theorem 7.1.6, their meaning, and why they are satisfied when $\mathcal{F}$ is a derived stack which is almost of finite presentation over R.

The necessity of condition (1) follows from Proposition 5.3.10. We note that condition (1) of Theorem 7.1.6 is weaker than the obvious analogue of the corresponding assumption in Theorem 7.1.1, which would require $\mathcal{F}$ to commute with all filtered colimits. The reason is that there are natural examples of moduli spaces which are not locally of finite presentation, to which we would like our theorem to apply. An example is the derived Hilbert scheme, which is not locally of finite presentation at points which classify subvarieties of projective space which are not local complete intersections. However, as we shall see, the derived Hilbert scheme is almost of finite presentation everywhere, so we can establish its existence using Theorem 7.1.6.

Condition (2) is of course satisfied by definition if $\mathcal{F}$ is *any* derived stack. Condition (3) is the obvious analogue of the corresponding condition in Theorem 7.1.1. One might also consider a derived analogue of the corresponding condition, using the derived formal geometry of the last section. This alternative formulation turns out to be equivalent, assuming that conditions (4) through (6) are satisfied:

Proposition 7.1.7. *Let $\mathcal{F} : \mathcal{SCR} \to \mathcal{S}$ be a nilcomplete, infinitesimally cohesive functor with a cotangent complex, and let $k \geq 0$. The following conditions are equivalent:*

- $(3'_k)$ *Let $B \in \mathcal{SCR}$ be complete, local, and Noetherian, and let $\mathfrak{m} \subseteq \pi_0 B$ denote the maximal ideal. Let $\{B_n\}$ be a pro-system representing the $\mathfrak{m}$-adic completion of B. Then the natural map $\mathcal{F}(\tau_{\leq k} B) \to \lim\{\mathcal{F}(\tau_{\leq k} B_n)\}$ is an equivalence.*

- (3′′) *Let $B \in \mathcal{SCR}$ be complete, local, and Noetherian. Then the natural map $\mathcal{F}(B) \to \lim\{\mathcal{F}(B_\alpha)\}$ is an equivalence, where the pro-system $\{B_\alpha\}$ represents the pro-Artinian completion of B.*
- (3′′′) *Let $B \in \mathcal{SCR}$ be complete, local, and Noetherian, with maximal ideal $\mathfrak{m} \subseteq \pi_0 B$. Let $\{B_n\}$ be a pro-system representing the $\mathfrak{m}$-adic completion of B. Then the natural map $\mathcal{F}(B) \to \lim\{\mathcal{F}(B_n)\}$ is an equivalence.*

Proof. We first show that the conditions $(3'_k)$ and $(3'_{k'})$ are equivalent for all $k, k' \geq 0$. It will suffice to treat the case where $k' = k+1$. Let $C = \tau_{\leq k} B$, and $C' = \tau_{\leq k+1} B$. Then C' is a square-zero extension of C by $M[k+1]$, for some discrete, finitely generated $\pi_0 B$-module M. We first prove that $(3'_k)$ implies $(3'_{k+1})$. For this, it suffices to show that for any point $\eta \in \mathcal{F}(C) \simeq \lim\{\mathcal{F}(\tau_{\leq k} B_n)\}$, the fibers F and F' of the natural maps $\mathcal{F}(C') \to \mathcal{F}(C)$ and $\lim\{\mathcal{F}(\tau_{\leq k+1} B_n)\} \to \lim\{\mathcal{F}(\tau_{\leq k} B_n)\}$ over the point η are equivalent. We note that F is nonempty if and only if a certain obstruction in $\pi_k \operatorname{Hom}_{\mathcal{M}_C}(L_{\mathcal{F}/R}(\eta), M)$ vanishes. Similarly, the nontriviality of F' is equivalent to the vanishing of an element in $\pi_k K$, where K is the $\mathfrak{m}$-adic completion of $\operatorname{Hom}_{\mathcal{M}_C}(L_{\mathcal{F}/R}(\eta), M)$. Since B is complete and M is coherent, M is $\mathfrak{m}$-adically complete so that $\operatorname{Hom}_{\mathcal{M}_C}(L_{\mathcal{F}/R}(\eta), M)$ is complete.

In the event that both are nonempty, one notes that F is a torsor for the space $\operatorname{Hom}_{\mathcal{M}_C}(L_{\mathcal{F}/R}(\eta), M[k+1])$, while F' is a torsor for the 0th space of the $\mathfrak{m}$-adic completion $\operatorname{Hom}_{\mathcal{M}_C}(L_{\mathcal{F}/R}(\eta), M[k+1])$. Since this C-module is already complete, we deduce that $F \simeq F'$.

We now prove that $(3'_{k+1}) \Rightarrow (3'_k)$. Using the above argument, we again deduce that $F \simeq F'$. Thus, if $\mathcal{F}(C') \simeq \lim\{\mathcal{F}(\tau_{\leq k+1} B)\}$, then the natural map $p : \mathcal{F}(C) \to \lim\{\mathcal{F}(\tau_{\leq k} B_n)\}$ is an inclusion of connected components, and it will suffice to show that p is surjective. For this, we are free to replace B by C and thereby assume that B is k-truncated. Then $\mathcal{F}(C) = \mathcal{F}(C') = \lim\{\mathcal{F}(\tau_{\leq k+1} B_n)\}$, and it suffices to prove that the natural map $\lim\{\mathcal{F}(\tau_{\leq k+1} B_n)\} \to \lim\{\mathcal{F}(\tau_{\leq k} B_n)\}$ is an equivalence. The proof of this is similar to the argument given above: since $\mathcal{F}$ is cohesive, the mapping fiber is controlled by the cotangent complex. We leave the details to the reader.

If B is n-truncated, the conditions (3′′) and (3′′′) are equivalent, since the $\mathfrak{m}$-adic completion of B is pro-equivalent to the pro-Artinian completion of B (see Remark 6.2.11). Moreover, if the maximal ideal of B may be generated by m elements, then we may guarantee that each B_n appearing an inverse system which represents that $\mathfrak{m}$-adic completion of B is $(n+m)$-truncated. Thus, (3′′) is also equivalent to $(3'_k)$ for $k \geq n+m$. By the first part of the proof, we see that (3′′) is equivalent to $(3'_k)$ for any $k \geq 0$.

The general case now follows from the fact that $\mathcal{F}$ is nilcomplete, using the equivalence $B \simeq \lim\{\tau_{\leq n} B\}$. □

Condition (3) of Theorem 7.1.6 is identical with assertion $(3'_0)$ of Proposition 7.1.7. Theorem 6.4.1 establishes that the equivalent conditions of Proposition 7.1.7 are satisfied in the case where $\mathcal{F}$ is a derived stack.

The necessity of condition (4) was established as Theorem 5.1.5. We have already remarked that conditions (4) and (5) are the natural analogue of Artin's condition (4) on the existence of obstruction and deformation theories for the functor $\mathcal{F}$.

Remark 7.1.8. If $L_{\mathcal{F}/R}$ exists, then condition (1) implies that $L_{\mathcal{F}/R}$ is almost perfect. To prove this, it suffices to show that for any map $\operatorname{Spec} B \to \mathcal{F}$, the functor $\operatorname{Hom}_{\mathcal{M}_B}(L_{\mathcal{F}/R}|\operatorname{Spec} B, \bullet)$ commutes with filtered colimits when restricted n-truncated objects of $\mathcal{M}_B$. Since $L_{\mathcal{F}/R}$ is almost connective, we can reduce to evaluating this functor on *connective* modules, in which case the commutativity with filtered colimits follows from condition (1) and the definition of the cotangent complex.

Conversely, if we assume that $\mathcal{F}$ satisfies (4) and (5) and that $L_{\mathcal{F}/R}$ is almost perfect, then (1) is equivalent to the (*a priori* weaker) assumption that $\mathcal{F}$ commutes with filtered colimits when restricted to *discrete* R-algebras. The proof is analogous to that of Proposition 7.1.7.

The nilcompleteness in condition (6) of Theorem 7.1.6 has no parallel in Theorem 7.1.1; the idea that an R-algebra B should be well-approximated by its truncations is unique to the derived context. The necessity of this condition (together with condition (5)) has been established in Proposition 5.4.5. Finally, we note that condition (7) is obviously necessary for the representability of $\mathcal{F}$ by an n-stack.

One difference between Theorem 7.1.1 and Theorem 7.1.6 is that the latter requires no hypothesis of relative representability for the functor $\mathcal{F}$. Roughly speaking, this is because all of the assumptions of 7.1.6 are sufficiently natural that they are preserved under passage to finite limits. One can therefore repeat the argument for the representability theorem, applied to the diagonal map $\mathcal{F} \to \mathcal{F} \times_{\operatorname{Spec} R} \mathcal{F}$ rather than $\mathcal{F} \to \operatorname{Spec} R$.

Let us now summarize the contents of this section. We will begin in §7.2 by proving the derived analogue of Artin's algebraization lemma. When combined with Theorem 6.2.13, we will be able to deduce that any functor $\mathcal{F}$ satisfying the hypotheses of Theorem 7.1.6 admits a smooth covering by a derived scheme. If $\mathcal{F}$ were relatively representable, we would be able to conclude the proof there. However, we will eliminate this assumption by iterating the argument.

Finally, in §7.4, we show how condition (4), the most mysterious of the hypotheses of Theorem 7.1.6, may be stated in more concrete terms. This leads to a reformulation of Theorem 7.1.6, which we shall state in §7.5.

Remark 7.1.9. From Theorem 7.1.6 one can also deduce criteria for the functor $\mathcal{F}$ to be representable by more specific sorts of geometric objects. For example, once we know that $\mathcal{F}$ is representable by a derived stack, it is representable by a derived scheme if and only if the cotangent complex $L_{\mathcal{F}/R}$ is connective. This derived scheme is a derived algebraic space if and only if $\mathcal{F}(A)$ is discrete for any discrete ring A. A derived stack $\mathcal{F}$ is locally of finite presentation over $\operatorname{Spec} A$ if and only if it satisfies a stronger version of condition (1), in which one considers arbitrary filtered colimits in $\mathcal{SCR}$.

7.2 Algebraization of Versal Deformations

Throughout this section, we will suppose that R is a derived G-ring and $\mathcal{F} : \mathcal{SCR} \to \mathcal{S}$ is a functor equipped with a natural transformation $\mathcal{F} \to \operatorname{Spec} R$.

Our goal is to represent $\mathcal{F}$ by a geometric object. As a first step towards doing so, we wish to produce a formally smooth morphism $\operatorname{Spec} C \to \mathcal{F}$ for some R-algebra C. It will be hard to get C right on the first try. Our first lemma asserts that, if we begin with a morphism $\operatorname{Spec} C_0 \to \mathcal{F}$ which is "almost" formally smooth (in the sense that part of the cotangent complex $L_{C_0/\mathcal{F}}$ vanishes at some point), then we can modify C_0 to obtain a morphism which is formally smooth.

Lemma 7.2.1. *Suppose that $\mathcal{F}$ satisfies conditions* (1), (4), (5), *and* (6) *of Theorem 7.1.6. Let $\eta : \operatorname{Spec} C_0 \to \mathcal{F}$ be a map, where C_0 has residue field k at some prime ideal of $\pi_0 C_0$. Suppose further that $\pi_1(L_{C_0/\mathcal{F}} \otimes_{C_0} k) = 0$. Then, possibly after shrinking C_0 to some Zariski neighborhood of* $\operatorname{Spec} k$, *there exists a factorization* $\operatorname{Spec} C_0 \to \operatorname{Spec} C \to \mathcal{F}$, *where $\tau_{\leq 0} C_0 \simeq \tau_{\leq 0} C$ and $\operatorname{Spec} C \to \mathcal{F}$ is formally smooth. Furthermore, if C_0 is Noetherian, then each $\pi_i C$ is a finitely generated C_0 module. Thus, if C_0 is almost of finite presentation over R, then so is C.*

Proof. We first note that conditions (1) and (4) imply that $L_{C_0/\mathcal{F}}$ exists and is almost perfect, so that we may choose an integer $n \leq 0$ such that $\pi_j L_{C_0/\mathcal{F}} = 0$ for $j < n$.

We will construct the map $\operatorname{Spec} C \to \mathcal{F}$ as the limit of a sequence of maps $\operatorname{Spec} C_i \to \mathcal{F}$, satisfying $\tau_{\leq i} C_i = \tau_{\leq i} C_{i+1}$. Moreover, we shall have

$$\pi_j(L_{C_i/\mathcal{F}} \otimes_{C_i} k) = 0$$

for $0 < j \leq i+1$.

Assume that C_i has been constructed, and let $M_i = L_{C_i/\mathcal{F}}$. The exact triangle

$$L_{C_i/\mathcal{F}} \otimes_{C_i} C_0 \to L_{C_0/\mathcal{F}} \to L_{C_0/C_i}$$

and assumption that $C_i \to C_0$ is an almost finitely presented surjection imply that M_i is almost perfect, and that $\pi_j M_i = 0$ for $j < n$.

The first step is to construct a triangle

$$K_i \to M_i \to N_i$$

where K_i is the dual of a connective, perfect complex and $\pi_j N_i = 0$ for $j \leq i+1$. In fact, we will do this so that K_i has a finite composition series by C_i-modules having the form $C_i[j]$, for $n \leq j \leq 0$. We give two constructions of this triangle. Our first construction works in general, but requires us to localize C_0. Since we are only free to localize C_0 finitely often, we give a second construction which does not require localization, but which works only for sufficiently large i.

Construction (1) : We first construct, for each $j \leq 0$, morphisms $\phi_i^j : K_i^j \to M_i$ such that the induced map on homotopy groups $\pi_m(K_i^j \otimes_{C_i} k) \to \pi_m(M_i \otimes_{C_i} k)$ is an isomorphism for $m \leq j$. The construction is by ascending induction on j, starting with $j = n-1$, where we may take $K_i^j = 0$. Supposing that K_i^j and ϕ_i^j have already been constructed for $j < 0$, let $P = \operatorname{coker}(\phi_i^j)$. Since $P \otimes_{C_i} k$ is j-connected, we may after localizing C_i suppose that

P itself is j-connected. It follows that $\pi_{j+1}(P \otimes_{C_i} k) = \mathrm{Tor}_0^{\pi_0 C_i}(\pi_{j+1} P, k)$, so that we may choose a finite collection of elements of $\pi_{j+1} P$ whose images form a basis for the k-vector space $\pi_{j+1}(P \otimes_{C_i} k)$. This collection of homotopy classes induces map $Q[j] \to P[-1]$, where Q is a finitely generated free C_i-module. Let K_i^{j+1} denote the cokernel of the composition $Q[j] \to P[-1] \to K_i^{j+1}$. Since the induced map $Q[j] \to K_i^j \to M_i$ factors through the composition $P[-1] \to K_i^j \to M_i$, it is zero, so we get a factorization of ϕ_i^j through some $\phi_i^{j+1} : K_i^{j+1} \to M_i$ having the desired property. Moreover, K_i^{j+1} is an extension of $Q[j+1]$ by K_i^j.

Now set $K_i = K_i^0$. By construction, K_i is a successive extension of C_i-modules of the form $C_i[j]$ for $n \leq j \leq 0$. Let N_i be the cokernel of ϕ_i^0. Then we have a long exact sequence

$$\ldots \to \pi_m(K_i \otimes_{C_i} k) \to \pi_m(M_i \otimes_{C_i} k) \to \pi_m(N_i \otimes_{C_i} k) \to \pi_{m-1}(K_i \otimes_{C_i} k).$$

By assumption, the homotopy groups $\pi_m(M_i \otimes_{C_i} k)$ vanish for $0 < m \leq i+1$, and are therefore isomorphic to $\pi_m(K_i \otimes_{C_i} k)$ for all $m \leq i+1$. Consequently, we deduce that $\pi_m(N_i \otimes_{C_i} k) = 0$ for all $m \leq i+1$. Since N_i is almost perfect, we may (after passing to a localization) suppose that $\pi_m N_i$ vanishes for $m \leq i+1$.

Before giving the second construction of K_i, we explain how to complete the construction of C_{i+1} from C_i. Consider the associated map $L_{C_i/\mathcal{F}} \to N_i$. This map classifies a square-zero extension of C_i by $N_i[-1]$ which we shall denote by C_{i+1}. By construction, C_{i+1} comes equipped with a map $\mathrm{Spec}\, C_{i+1} \to \mathcal{F}$. Since $N_i[-1]$ is i-connected, the morphism $C_{i+1} \to C_i$ is $(i+1)$-connected, so the induced map $\tau_{\leq i} C_{i+1} \to C_i$ is an equivalence. Moreover, by Theorem 3.2.16, we have a natural $(i+3)$-connected morphism $N_i \otimes_{C_{i+1}} C_i \to L_{C_i/C_{i+1}}$. Composing with the $(2i+3)$-connected morphism $N_i \to N_i \otimes_{C_{i+1}} C_i$, we deduce that $N_i \to L_{C_i/C_{i+1}}$ is $(i+3)$-connected.

Now we note that $M_{i+1} \otimes_{C_{i+1}} k$ is the kernel of the natural map $M_i \otimes_{C_i} k \to L_{C_i/C_{i+1}} \otimes_{C_i} k$. Moreover, the connectivity estimate above implies that the induced map on π_m is an isomorphism for $0 < m < i+3$ and a surjection for $m = i+3$. Consequently, we deduce that $\pi_m M_{i+1} \otimes_{C_{i+1}} k$ vanishes for $0 < m \leq i+2$, as desired.

Construction (2) : Assume $i > (1-n)$. We suppose also that the triangle

$$K_{i-1} \to M_{i-1} \to N_{i-1}$$

has already been constructed. Our first goal is to construct a C_i-module K_i, together with an equivalence $K_{i-1} \otimes_{C_i} C_{i-1}$. For this, we make use of the fact that K_{i-1} admits a filtration by shifts of free modules. More precisely, we have

$$0 = K_{i-1}^{n-1} \to \ldots \to K_{i-1}^0 = K_{i-1},$$

and each K_{i-1}^{j+1} is obtained as the cokernel of some map $Q[j] \to K_{i-1}^j$, where Q is free. In order to lift K_{i-1}^{j+1} to a C_i-module K_{i-1}^j, it suffices to lift K_{i-1}^j and to lift the corresponding generators in $\pi_j K_{i-1}^j$. In other words, we need only know that the corresponding map $\pi_j K_i^j \to \pi_j K_i^j$ is surjective. For this, we need only know that $\pi_j(K_i^j \otimes_{C_i} T) = 0$, where T

denotes the cokernel of $C_i \to C_{i-1}$. Since K_i^j is $(n-1)$-connected, this is possible whenever T is $(j-n)$-connected: this follows from the inequality $i > (1-n)$.

Supposing that the C_i-module K_i has been constructed, we may interpret the C_{i-1}-module morphism $K_{i-1} \to M_{i-1}$ as a C_i-module morphism $f : K_i \to M_{i-1}$. Since $M_i \to M_{i-1}$ is $(i-1)$-connected and K_i is constructed out of cells having dimension < 0, the map f factors through some $f' : K_i \to M_i$; let N_i denote the cokernel of f'. We wish to show that N_i is $(i+1)$-connected. It suffices to prove this after tensoring with C_{i-1}. By the octahedral axiom, we have a triangle

$$N_i \otimes_{C_i} C_{i-1} \to N_{i-1} \to L_{C_{i-1}/C_i}.$$

As observed above, the natural map $N_{i-1} \to L_{C_{i-1}/C_i}$ is $(i+2)$-connected, so that N_i is $(i+1)$-connected as desired.

Now let C denote the inverse limit of the increasingly connected tower $\{C_i\}$. Since $\mathcal{F}$ is nilcomplete, we may choose a map $\operatorname{Spec} C \to \mathcal{F}$ which compatibly factors each of the maps $\operatorname{Spec} C_i \to \mathcal{F}$ that we have constructed. It is easy to see that $L_{C/\mathcal{F}}$ is the dual of a connective, perfect complex (in fact, it is the inverse limit of the tower $\{K_i\}$, which we have chosen compatibly for $i \gg 0$). □

The goal of this section is to prove the following result:

Lemma 7.2.2. *If the functor $\mathcal{F}$ satisfies conditions* (1) *through* (6) *of Theorem 7.1.6, then there exists a formally smooth surjection $U \to \mathcal{F}$, where U is a derived scheme almost of finite presentation over* $\operatorname{Spec} R$.

Proof. The construction of U is simple: we take U to be the disjoint union of $\operatorname{Spec} A$, indexed by all (equivalence classes of) formally smooth morphisms $\operatorname{Spec} A \to \mathcal{F}$ with source almost of finite presentation over $\operatorname{Spec} R$. The only nontrivial point is to verify that $U \to \mathcal{F}$ is a surjection of étale sheaves. In fact, we will show more: that $U \to \mathcal{F}$ is a surjection of Nisnevich sheaves.

Consider any map $\operatorname{Spec} A \to \mathcal{F}$. We must show that, locally on $\operatorname{Spec} A$, this map factors through U. Since U is formally smooth over $\mathcal{F}$, we may (using Proposition 3.4.5) reduce to the case where A is discrete. Using condition (1), we may suppose that A is almost of finite presentation over R, and therefore Noetherian.

Choose any prime ideal $\mathfrak{p}$ of A; we must show that there exists a factorization $\operatorname{Spec} A \to U$ in some neighborhood of $\mathfrak{p}$. Using condition (1) again, we may replace A by its Henselization at $\mathfrak{p}$.

Let k denote the residue field of A, and $\eta_0 \in \mathcal{F}(k)$ the associated element. Let $\mathcal{C}_k$ denote the ∞-category of Artinian local objects of $\mathcal{SCR}$ having residue field identified with k as in §6.2, and let $\mathcal{F}_0 : \mathcal{C}_k \to \mathcal{S}$ denote the functor given by the fiber of $\mathcal{F}$ over η_0.

The map induced map $R \to k$ determines a formally cohesive functor $\mathcal{F}_R$ on $\mathcal{C}_k$ and a natural transformation $\mathcal{F}_0 \to \mathcal{F}_R$. It follows that $\mathcal{F}_0$ is also a formally cohesive functor and that the homotopy groups of $T_{\mathcal{F}_0}$ are finite dimensional k-vector spaces in each degree. Theorem 6.2.13 now implies that there exists a complete Noetherian local $R' \in \mathcal{SCR}$ having residue field k and a formally smooth transformation $\mathcal{F}_{R'} \to \mathcal{F}_0$. (In the argument that follows, we will only actually use the fact that $\pi_1 T_{\mathcal{F}_{R'}/\mathcal{F}_0} = 0$. Consequently, we could replace R' by $\pi_0 R'$, which could be produced using the classical version of Schlessinger's criterion.)

The natural transformation $\mathcal{F}_0 \to \mathcal{F}_R$ induces an R-algebra structure on R' (compatible with the R-algebra structure on k). Using conditions (3) and (5), we deduce that the transformation $\mathcal{F}_{R'} \to \mathcal{F}_0$ uniquely determines a point $\hat{\eta} \in \mathcal{F}(R')$ lifting $\eta_0 \in \mathcal{F}(k)$.

Using the formal versality of $\hat{\eta}$, we may construct a transformation $f : R' \to \hat{A}$ over k together with an equivalence between the composite $\operatorname{Spec} \hat{A} \to \operatorname{Spec} A \to \mathcal{F}$ and $f^*\hat{\eta}$. Let us suppose known the existence of a factorization $\operatorname{Spec} R' \to \operatorname{Spec} R'_0 \to \mathcal{F}$, where R'_0 is formally smooth over $\mathcal{F}$ and almost of finite presentation over R. By Popescu's theorem, we may write $\hat{A}$ as a filtered colimit of smooth A-algebras A_α (equipped with distinguished k-valued points). The composite map $R'_0 \to R' \to \hat{A}$ therefore factors through A_α for sufficiently large α. Using condition (1), we may suppose (enlarging α if necessary) that the maps $\operatorname{Spec} A_\alpha \to \operatorname{Spec} A \to \mathcal{F}$ and $\operatorname{Spec} A_\alpha \to \operatorname{Spec} R'_0 \to \mathcal{F}$ are homotopic. Since A_α is smooth over A and the closed fiber of $\operatorname{Spec} A_\alpha$ has a rational point over k, the assumption that A is Henselian implies the existence of a section $A_\alpha \to A$, which proves that the original map $\operatorname{Spec} A \to \mathcal{F}$ factors through R'_0.

It remains to construct R'_0 (in other words, we have reduced ourselves to the case where $A = R'$). This is the main point of the proof. The morphism $\operatorname{Spec} R' \to \mathcal{F}$ is formally versal, but R' is not almost of finite presentation over R. We wish to find an approximation R'_0 to R' which is almost of finite presentation and still versal (that is, formally smooth) over $\mathcal{F}$. This is usually done by algebraizing R': that is, choosing R'_0 such that R' is the completion of some localization of R'_0. We will give a simpler argument which tells us a little bit less: it shows only that the completion of R'_0 is (infinitesimally) formally smooth over R'. However, this will be enough to complete the proof, since it will imply that $R'_0 \to \mathcal{F}$ is formally smooth at the point in question.

We begin by noting that k is finitely generated (as a field) over $\pi_0 R$. Consequently, we can find a factorization $R \to B \to R'$, where $B = R[x_1, \ldots, x_m]$, and B has residue field k at some point $\mathfrak{r} \subseteq \pi_0 B$ lying over the maximal ideal of R'.

Enlarging B if necessary, we may suppose that $\mathfrak{r}/\mathfrak{r}^2$ surjects onto the Zariski tangent space of $\pi_0 R'$. Let $\hat{B}$ denote the completion of $B_{\mathfrak{r}}$ at its maximal ideal, so that the induced map $f : \hat{B} \to R'$ is surjective. Consequently, the kernel K of f is an almost perfect $\hat{B}$-module; choose a surjection $\beta : \hat{B}^n \to K$.

By Theorem 3.7.5, $\hat{B}$ is a filtered colimit of smooth B-algebras $\{B_\alpha\}$. Let K_α denote the kernel of the composite map $B_\alpha \to \hat{B} \to R'$. Then K is the filtered colimit of $\{K_\alpha\}$, so that for sufficiently large α, the map β factors through some map $\widetilde{\beta} : B_\alpha^n \to K_\alpha$. Let $C_{\widetilde{\beta}}$ denote the $B_\alpha \otimes_{\operatorname{Sym}^*_B B^n} B$ denote the B_α algebra obtained by killing the image of $\widetilde{\beta}$. We note that the liftings $\widetilde{\beta}$ (as α varies) form a filtered ∞-category, so that the algebras $\{C_{\widetilde{\beta}}\}$ form a filtered system with colimit $C = \hat{B} \otimes_{\operatorname{Sym}^*_{\hat{B}} \hat{B}^n} \hat{B}$. We also note that by construction, the $\{C_{\widetilde{\beta}}\}$ come equipped with a compatible family of maps to R', whose colimit is the natural map $C \to R'$.

By construction, $C \to R'$ is 1-connected. Let K' denote the kernel of $C[-1] \to R'[-1]$. We now repeat the above construction to further improve the connectivity. The C-module K' is connective and almost perfect, so that there exists a surjection $\gamma : C^m \to K'$. Let $K'_{\widetilde{\beta}}$ denote the kernel of $C[-1]_{\widetilde{\beta}} \to R'[-1]$. Arguing as above, the collection of all choices $\widetilde{\beta}$ together with factorizations $\widetilde{\gamma} : C^m_{\widetilde{\beta}} \to K'$ form a filtered collection, giving rise to a filtered system $\{D_{\widetilde{\gamma}}\} = \{C_{\widetilde{\beta}} \otimes_{\mathrm{Sym}^*_{C_{\widetilde{\beta}}} C^m_{\widetilde{\beta}}[1]} C_{\widetilde{\beta}}\}$ having filtered colimit D equipped with a 2-connected map $D \to R'$.

In particular, $\tau_{\leq 1} R'$ is the filtered colimit of the system $\{\tau_{\leq 1} D_{\widetilde{\gamma}}\}$. Consequently, condition (1) implies that $\hat{\eta}|\tau_{\leq 1} R'$ is the image of some element of $\mathcal{F}(\tau_{\leq 1} D_{\widetilde{\gamma}})$ for sufficiently large $\widetilde{\gamma}$. To simplify the notation, we shall henceforth write D' for $D_{\widetilde{\gamma}}$, D_0 for $\tau_{\leq 1} D$, and D'_0 for $\tau_{\leq 1} D'$.

Let $M = L_{D'_0/\mathcal{F}} \otimes_{D_0} k$. We claim that $\pi_1 M = 0$. Granting this for the moment, let us show how to finish the proof of the Lemma. Applying Lemma 7.2.1 to the morphism $\operatorname{Spec} D'_0 \to \mathcal{F}$, we deduce the existence of a factorization $\operatorname{Spec} D'_0 \to \operatorname{Spec} D'' \to \mathcal{F}$, where $\operatorname{Spec} D''$ is formally smooth over $\mathcal{F}$, almost of finite presentation over R, and the associated morphism $D'' \to D'_0$ induces an isomorphism on π_0. By assumption, we have a factorization $\operatorname{Spec} \tau_{\leq 0} R' \xrightarrow{p} \operatorname{Spec} D'_0 \to \mathcal{F}$. Applying Proposition 3.4.5, we conclude the proof.

It remains to prove that $\pi_1 M = 0$. To prove this, we make use of the exact triangle

$$M \to L_{D_0/\mathcal{F}} \otimes_{D_0} k \to L_{D_0/D'_0} \otimes_{D_0} k.$$

It suffices to prove that $\pi_1(L_{D_0/\mathcal{F}} \otimes_{D_0} k) = 0$ and $\pi_2(L_{D_0/D'_0} \otimes_{D'_0} k) = 0$.

To show that $\pi_1(L_{D_0/\mathcal{F}} \otimes_{D_0} k) = 0$, we note that $D_0 \simeq \tau_{\leq 1} R'$, and use the exact triangle

$$L_{R'/\mathcal{F}} \otimes_{R'} k \to L_{\tau_{\leq 1} R'/\mathcal{F}} \otimes_{\tau_{\leq 1} R'} k \to L_{\tau_{\leq 1} R'/R'} \otimes_{\tau_{\leq 1} R'} k.$$

It now suffices to prove that $\pi_1(L_{R'/\mathcal{F}} \otimes_{R'} k) = 0$ and $\pi_1(L_{\tau_{\leq 1} R'/R'} \otimes_{\tau_{\leq 1} R'} k) = 0$. The first part follows from the fact that $\mathcal{F}_{R'} \to \mathcal{F}_0$ is formally smooth, and the second follows from Theorem 3.2.16.

We now prove that $\pi_2(L_{D_0/D'_0} \otimes_{D_0} k) = 0$. By Theorem 3.2.16, we have $\pi_1(L_{D'_0/D'} \otimes_{D'_0} k) = 0$. Using the exact triangle

$$L_{D'_0/D'} \otimes_{D'_0} D_0 \to L_{D_0/D'} \to L_{D_0/D'_0}$$

we see that it suffices to prove that $\pi_2(L_{D_0/D'} \otimes_{D_0} k) = 0$. Applying Theorem 3.2.16 again, we deduce that $\pi_2(L_{D_0/D} \otimes_{D_0} k) = 0$. Using the exact triangle

$$L_{D/D'} \otimes_D D_0 \to L_{D_0/D'} \to L_{D_0/D}$$

we may reduce to proving that $\pi_2(L_{D/D'} \otimes_D k) = 0$.

Let $\mathfrak{q}'$ denote the preimage of the maximal ideal of $\pi_0 R'$ in D', and let $\hat{D}'$ denote the completion of the localization of D' at $\mathfrak{q}'$. Similarly, one may define $\hat{B}'$ as a completion of B_γ. We note that $D \simeq \hat{B} \otimes_{\hat{B}'} \hat{D}'$. We note that $L_{D/D'} \otimes_D k$ does not change if we replace D'

by its completion $\hat{D}'$. Consequently, we deduce that $L_{D/D'} \otimes_D k \simeq L_{\hat{B}/\hat{B}'} \otimes_{\hat{B}} k \simeq L_{\hat{B}/B_\gamma} \otimes_{\hat{B}} k$. Now we use the exact triangle

$$L_{B_\alpha/B} \otimes_{B_\alpha} k \to L_{\hat{B}/B} \otimes_{\hat{B}} k \to L_{\hat{B}/B_\alpha} \otimes_{\hat{B}} k.$$

We note that the middle term vanishes, so that the associated long exact sequence degenerates to give an isomorphism $\pi_2(L_{\hat{B}/B_\gamma} \otimes_{\hat{B}} k) \simeq \pi_1(L_{B_\gamma/B} \otimes_{B_\gamma} k)$. This latter group vanishes since B_γ is smooth over B. □

Remark 7.2.3. The proof of Lemma 7.2.2 actually shows that $U \to \mathcal{F}$ is a surjection for the *Nisnevich* topology. It follows that if $\mathcal{F}$ is a functor satisfying (1) through (6), then we may choose a hypercovering $U_\bullet$ of $\mathcal{F}$ by Nisnevich derived schemes (this hypercovering has geometric realization $\mathcal{F}$ if A has finite Krull dimension, by Theorem 4.4.5). Consequently, if $T : \mathcal{SCR} \to \mathcal{S}$ is a sheaf for the Nisnevich topology, we may set $T(\mathcal{F}) = |T(U_\bullet)|$ to get a reasonable definition of T for a large class of derived stacks. This seems to give a plausible definition for the higher Chow groups of stacks, which compares well with other definitions for quotients of quasi-projective varieties by linear algebraic group actions. We will discuss this point in greater detail in [23].

7.3 Proof of the Representability Theorem

The goal of this section is to give the proof of Theorem 7.1.6. We remark that, if the functor $\mathcal{F}$ satisfies the hypotheses of Theorem 7.1.6 and $\mathcal{F} \to \mathcal{F} \times_{\operatorname{Spec} A} \mathcal{F}$ is known to be a relative stack which is almost of finite presentation, then the conclusion follows immediately from the definition of a relative stack and Lemma 7.2.2. The reader who is satisfied with this slightly weaker version of the representability theorem may skip this section, which is devoted to removing the relative representability hypothesis.

Let A be a derived G-ring. Let us call a functor $\mathcal{F} \in \operatorname{Shv}(\mathcal{SCR}_{/A}^{op})$ *n-good* if it satisfies the hypotheses (1) through (7) of Theorem 7.1.6. We have already established that any derived n-stack which is almost of finite presentation over $\operatorname{Spec} A$ is an n-good functor, and we wish to prove the converse. We note that the class of n-good functors is stable under finite limits (in the ∞-category of functors *over* $\operatorname{Spec} A$).

Suppose that $\mathcal{F}$ is an n-good functor. Then by Lemma 7.2.2 there exists a smooth surjection of étale sheaves $U \to \mathcal{F}$, where U is disjoint union of affine derived schemes which are almost of finite presentation over A. To complete the proof, it suffices to show that $U \to \mathcal{F}$ is a relative stack which is almost of finite presentation. In other words, we must show that $\operatorname{Spec} R \times_{\mathcal{F}} U$ is a derived stack, almost of finite presentation over R, for any morphism $\operatorname{Spec} R \to \mathcal{F}$. This assertion is local on $\operatorname{Spec} R$, so we may assume the existence of a factorization $\operatorname{Spec} R \to U \to \mathcal{F}$ and thereby replace $\operatorname{Spec} R$ by U. Now we note that $U \times_{\mathcal{F}} U$ is another good functor. If $n > 0$, then $U \times_{\mathcal{F}} U$ is $(n-1)$-good. Consequently, we may work by induction on n and reduce to the case where $n = 0$.

Since $\mathcal{F}$ takes discrete values on ordinary commutative rings, the relative cotangent complex $L_{\mathcal{F}/A}$ is *connective.* In this situation, we shall prove the following refinement of 7.2.2:

Lemma 7.3.1. *Suppose that $\mathcal{F}$ is a good functor and that $L_{\mathcal{F}/A}$ is connective. Then there exists a formally étale surjection $U' \to \mathcal{F}$, where U' is a disjoint union of affine derived schemes which are almost of finite presentation over A.*

Proof. We begin with the formally smooth surjection $U \to X$ provided by Lemma 7.2.2. We then define U' as in the proof of Theorem 5.1.12. Namely, we consider all instances of the following data: étale morphisms $\operatorname{Spec} R \to U$ together with m-tuples $\{r_1, \ldots, r_m\} \subseteq \pi_0 R$ such that $\{dr_1, \ldots, dr_m\}$ freely generate $\pi_0 L_{U/\mathcal{F}}(R)$. For each such tuple, we let R' denote the R-algebra obtained by killing (lifts of) $\{r_1, \ldots, r_m\}$. Let U' denote the derived scheme which is the disjoint union of $\operatorname{Spec} R'$, taken over all R' which are obtained in this way.

By construction, U' is almost of finite presentation over A and $\pi : U' \to \mathcal{F}$ is formally étale. To complete the proof, we need only show that π is surjective. The proof proceeds along the lines of the proof of Theorem 5.1.12, except for the obstacle that we may not assume that $U' \to \mathcal{F}$ is a relative derived scheme.

Choose any morphism $\operatorname{Spec} k \to \mathcal{F}$, where $k \in \mathcal{SCR}$. We wish to show that, étale locally on $\operatorname{Spec} k$, this map factors through U'. Since U' is formally étale over $\mathcal{F}$, we may reduce to the case where k is a discrete commutative ring. Since both $\mathcal{F}$ commutes with filtered colimits when restricted to discrete objects, we may suppose that B is finitely generated as a discrete $\pi_0 A$-algebra. Consequently, k is almost of finite presentation over A, so that $\operatorname{Spec} k$ is a good functor. It will suffice to show that the base chance $\pi' : \operatorname{Spec} k \times_{\mathcal{F}} U' \to \operatorname{Spec} k$ is a surjection of étale sheaves. Since $\operatorname{Spec} k \times_{\mathcal{F}} U'$ is a good functor, Lemma 7.2.2 implies the existence of a formally smooth surjection $V \to \operatorname{Spec} k \times_{\mathcal{F}} U'$ where V is a derived scheme which is almost of finite presentation over A. The composite map $V \to \operatorname{Spec} k$ is a formally smooth morphism of derived schemes which are almost of finite presentation over R. Consequently, V is smooth over $\operatorname{Spec} k$, so that the image of $V \to \operatorname{Spec} k$ is open in the Zariski topology of $\operatorname{Spec} k$. To complete the proof, it will suffice to prove that this image contains every closed point of $\operatorname{Spec} k$. Replacing k by its residue field at such a point, we may suppose that (as suggested by the notation) the commutative ring k is a field. We wish to show that, possibly after making a separable extension of k, the map $\operatorname{Spec} k \to \mathcal{F}$ factors through U'.

Since $U \to \mathcal{F}$ is surjective, we may (after making a separable extension of k) choose a factorization $\operatorname{Spec} k \to \operatorname{Spec} R \to U \to \mathcal{F}$, where $\operatorname{Spec} R$ is étale over U. The connectivity of $L_{\mathcal{F}}$ implies that $L_U \to L_{U/\mathcal{F}}$ is surjective. Localizing R if necessary, we may suppose that there exist $\{x_1, \ldots, x_m\} \subseteq \pi_0 R$ whose differentials generate $L_{U/\mathcal{F}}$. A choice of such elements gives rise to a formally étale morphism $\operatorname{Spec} R \to \mathbf{A}^m_{\mathcal{F}}$. Since $\operatorname{Spec} R \times_{\mathcal{F}} \operatorname{Spec} k$ is a good functor, Lemma 7.2.2 implies the existence of a formally smooth surjection $W \to \operatorname{Spec} R \times_{\mathcal{F}} \operatorname{Spec} k$, where W is a derived scheme almost of finite presentation over A. The composite map $W \to \mathbf{A}^m_{\mathcal{F}} \times_{\mathcal{F}} \operatorname{Spec} k = \mathbf{A}^m_k$ is a formally smooth morphism of derived schemes which are almost of finite presentation over k, hence smooth. Consequently, the image of W is a Zariski-open subset of $\mathbf{A}^m_k$. It follows that this image contains a point whose coordinates are all separably algebraic over the prime field of k. Passing to a separable extension of k if necessary, we may suppose that the coordinates of W all lie in a subfield $k_0 \subseteq k$ such that k_0 is a finite separable extension of the prime field of k. After passing to a separable extension of k, we may lift this to a k-valued point of W. This gives us a new factorization

$\operatorname{Spec} k \xrightarrow{f} \operatorname{Spec} R \to U \to \mathcal{F}$ having the additional property that each $f^* x_i \in k_0 \subseteq k$ for each of our coordinates $x_i \in \pi_0 R$.

Let R' denote the (Zariski) localization of R at the image point of f, and let $\mathfrak{m} \subseteq \pi_0 R'$ denote the maximal ideal. To prove that $\operatorname{Spec} k \to \mathcal{F}$ factors through U', it will suffice to show that we choose $\{r_1, \ldots, r_m\} \subseteq \mathfrak{m}$ such that the differentials $\{dr_1, \ldots, dr_m\}$ freely generate $\pi_0 L_{U/\mathcal{F}}(R')$. Since $\pi_0 L_{U/\mathcal{F}}(R')$ is free over $\pi_0 R'$, Nakayama's lemma implies that this is equivalent to the surjectivity of the natural map $\pi_1 L_{k/U} = \mathfrak{m}/\mathfrak{m}^2 \to \pi_0 L_{U/\mathcal{F}}(k) = \pi_0 L_{\mathbf{A}^m_{\mathcal{F}}/\mathcal{F}}(k)$. Using the long exact sequence, we see that this is equivalent to the assertion that the natural map $\pi_0 L_{\mathbf{A}^m_{\mathcal{F}}/\mathcal{F}}(k) \to \pi_0 L_{k/\mathcal{F}}$ is zero. Since $L_{\mathcal{F}}$ is connective, $\pi_0 L_{k/\mathcal{F}}$ is a quotient of $\pi_0 L_k = \Omega_{k/k_0}$. it therefore suffices to show that the differentials of each of the coordinate functions $\{x_1, \ldots, x_m\}$ vanish in Ω_{k/k_0}. This is clear, since the coordinates take their values in k_0 by construction. □

We are now prepared to give the proof of Theorem 7.1.6. Let $\mathcal{F}$ be a functor satisfying the hypotheses of the theorem. As explained above, we may reduce to the case where $\mathcal{F}$ is 0-good. Let us first treat the special case in which there exists a morphism $\mathcal{F} \to X$, where $X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a derived algebraic space which is almost of finite presentation over A, having the property that $\mathcal{F}(B) \to X(B)$ is an isomorphism (of sets) whenever B is discrete. We wish to show that $\mathcal{F}$ is representable by a derived scheme. Using Lemma 7.3.1, we may deduce the existence of a formally étale surjection $U \to \mathcal{F}$, where $U = (\mathcal{U}, \mathcal{O}_{\mathcal{U}})$ is a derived scheme almost of finite presentation over A. Then the induced map $\pi : \tau_{\leq 0} U \to \tau_{\leq 0} X$ is étale. The assertion that $\mathcal{F}$ is a derived scheme is local on X, so we may suppose that π admits a section s. It is easy to see that $(\mathcal{X}, s^* \mathcal{O}_{\mathcal{U}})$ is a derived scheme which represents the functor $\mathcal{F}$.

We now treat the slightly more general case in which there exists a morphism $\mathcal{F} \to X$ which induces an *injective* map of sets $\mathcal{F}(A) \to X(A)$ for any discrete commutative ring A, where X is a derived algebraic space which is almost of finite presentation over A. Choose a formally smooth surjection $U \to \mathcal{F}$, where U is a derived algebraic space which is almost of finite presentation over A. To show that $\mathcal{F}$ is a derived stack, it suffices to show that $U \times_{\mathcal{F}} U$ is a derived stack. But $U \times_{\mathcal{F}} U$ is a good functor and the map $U \times_{\mathcal{F}} U \to U \times_X U$ induces an isomorphism of sets when evaluated on any discrete commutative ring, so we deduce the desired result from the previous step.

Finally, let us consider the case where $\mathcal{F}$ is a general 0-good functor. Choose again a formally smooth surjection $U \to \mathcal{F}$, where U is a derived algebraic space which is almost of finite presentation over A. Once again, it will suffice to show that $U \times_{\mathcal{F}} U$ is a derived scheme. But the natural map $U \times_{\mathcal{F}} U \to U \times_{\operatorname{Spec} A} U$ induces an injection when evaluated on discrete commutative rings, so we again reduce to the previous step. This concludes the proof of Theorem 7.1.6.

7.4 Existence of the Cotangent Complex

Let $\mathcal{F} : \mathcal{SCR} \to \mathcal{S}$ be a functor, and $\mathcal{F} \to \operatorname{Spec} R$ a natural transformation. Theorem 7.1.6 asserts that $\mathcal{F}$ is representable by a geometric object provided that certain natural conditions are satisfied. Condition (4), the existence of a cotangent complex for $\mathcal{F}$, is perhaps the most subtle of these. The purpose of this section is to show that, under suitable conditions, we may reformulate condition (4) in more classical terms.

The idea is that if functor $\mathcal{F}$ is to have a cotangent complex $L_{\mathcal{F}}$, then the relative cotangent complex $L_{\mathcal{F}/R}$ will be almost perfect and we should be able to understand it in terms of its dual $T_{\mathcal{F}/R} = \operatorname{Hom}_{\mathrm{QC}_{\mathcal{F}}}(L_{\mathcal{F}/R}, \mathcal{O}_{\mathcal{F}})$. On the other hand, the dual should be easily characterized using the definition of the cotangent complex: if $\eta \in \mathcal{F}(C)$, then $T_{\mathcal{F}/R}(\eta)[j]$ should be essentially the same thing as the C-module $\mathcal{F}(C \oplus C[j]) \to \mathcal{F}(C) \times_{\operatorname{Hom}_{\mathcal{SCR}}(R,C)} \operatorname{Hom}_{\mathcal{SCR}}(R, C \oplus C[j])$. On the other hand, we can use this description to *define* $T_{\mathcal{F}/R}$, without assuming that $L_{\mathcal{F}/R}$ exists at all.

For the remainder of this section, we shall make the following assumptions:

1. For any discrete commutative ring A, the space $\mathcal{F}(A)$ is n-truncated.

2. The functor $\mathcal{F}$ is cohesive.

3. The functor $\mathcal{F}$ is nilcomplete.

4. The functor $\mathcal{F}$ is a sheaf for the étale topology.

5. The functor $\mathcal{F}$ commutes with filtered colimits when restricted to $\tau_{\leq k} \mathcal{SCR}$, for each $k \geq 0$.

6. $R \in \mathcal{SCR}$ is Noetherian and has a dualizing module.

Suppose that these assumptions are satisfied. Let $C \in \mathcal{SCR}$, let $\eta \in \mathcal{F}(C)$, and let M be a C-module. For $j \in \mathbf{Z}$, let $S_j(M)$ denote the mapping fiber of

$$\mathcal{F}(C \oplus (\tau_{\geq 0} M[j])) \to \mathcal{F}(C) \times_{\operatorname{Hom}_{\mathcal{SCR}}(R,C)} \operatorname{Hom}_{\mathcal{SCR}}(R, C \oplus (\tau_{\geq 0} M[j])).$$

Moreover, we have natural maps $\phi_j(M) : S_j(M) \to \Omega S_{j+1}(M)$. Since $\mathcal{F}$ is cohesive, $\phi_j(M)$ is an equivalence whenever $M[j]$ is connective.

Since $\mathcal{F}$ is takes n-truncated values on ordinary commutative rings, one can show that $\tau_{\geq m} \Omega^j S_j(M)$ is independent of j for $j \gg 0$. We let $S(M)$ denote the spectrum whose ith space is given by the colimit $\operatorname{colim} \Omega^j S_{i+j}(M)$. Finally, we set $T_i(\eta) = \pi_i S(C)$. This is a $\pi_0 C$-module which we may think of as an ith component of the tangent space to $\mathcal{F}$ at the point η. In concrete terms, we have $T_i(\eta) = \pi_{i+m} F_m$ for $m \geq 0, -i$, where F_m is the homotopy fiber of

$$\mathcal{F}(C \oplus C[m]) \to \mathcal{F}(C) \times_{\operatorname{Hom}_{\mathcal{SCR}}(R,C)} \operatorname{Hom}_{\mathcal{SCR}}(R, C \oplus C[m]).$$

We are now prepared to state the main result of this section, which was suggested to us by Bertrand Toën.

Theorem 7.4.1. *Suppose that $\mathcal{F}$ and R satisfy conditions* (1) *through* (6) *listed above. There exists an almost perfect cotangent complex $L_{\mathcal{F}/R}$ for $\mathcal{F}$ if and only if, for every discrete integral domain C, and every $\eta \in \mathcal{F}(C)$ which exhibits C as a finitely generated $\pi_0 R$-algebra, each of the tangent modules $T_j(\eta)$ is finitely generated over C.*

Proof. Suppose first that $L_{\mathcal{F}/R}$ exists and is almost perfect. Then $S_j(M) = \mathrm{Hom}_{\mathcal{M}_C}(L_{\mathcal{F}/R}, M[j])$ and is therefore a coherent C-module. It follows that each homotopy group of $S_j(M)$ is finitely generated as a discrete C-module.

To prove the converse, we begin by showing that if C is truncated, $\eta \in \mathcal{F}(C)$ exhibits C as an R-module which is almost of finite presentation, then the functor $M \mapsto S(M)$ satisfies the hypotheses of Theorem 3.6.9. Conditions (1), (2), and (4) are easy to verify. For condition (3), we may filter M and thereby reduce to the case where $M \simeq C_0 = \pi_0 C/\mathfrak{p}$, where $\mathfrak{p} \subseteq \pi_0 C$ is a prime ideal. Then $C \oplus M \simeq (C_0 \oplus M) \times_{C_0} C$. Using the fact that $\mathcal{F}$ is cohesive, we may reduce to the case where C is a discrete integral domain, in which case the finite generation follows from the hypothesis of the theorem.

Theorem 3.6.9 now implies the existence of an almost perfect C-module $L_{\mathcal{F}/R}(\eta)$ having the appropriate mapping property. We next show that if $p : C \to C'$ is almost of finite presentation and C' is also truncated, then the natural map $\psi_p : L_{\mathcal{F}/R}(\eta) \otimes_C C' \to L_{\mathcal{F}/R}(p_* \eta)$ is an equivalence. If p is surjective, then we simply use the universal mapping property of $L_{\mathcal{F}/R}$ and the assumption that $\mathcal{F}$ is cohesive. In the general case, we may consider a factorization $C \to C[x_1, \ldots, x_m] \to C'$ where the second map is surjective, and thereby reduce to the case where $C' = C[x_1, \ldots, x_m]$. Working by induction on m, we may reduce to the case where $C' = C[x]$.

Suppose that ψ_p is not an equivalence, and let K_p denote its kernel. Then K_p is an almost perfect $C[x]$-module, so there exists some smallest value of j such that $\pi_j K_p$ is nonzero. Then $\pi_j K_p$ is a finitely generated module over $\pi_0 C[x]$, whose formation is compatible with surjective base change. The module $\pi_j K_p$ has nonzero localization at some maximal ideal of $\pi_0 C[x]$, which lies over some maximal ideal $\mathfrak{m}$ of $\pi_0 C$. Replacing C by $\pi_0 C/\mathfrak{m}$, we may suppose that C is a field k.

For any element $a \in k$, the evaluation map $e : k[x] \to k$ which carries x into a is a surjection, and induces the identity map $k \to k$. Consequently, ψ_e and $\psi_{e \circ p}$ are equivalences, so that $K_p \otimes_{k[x]} k = 0$. Consequently, the $k[x]$-module $M = \pi_j K_p$ does not have support at any k-valued point of $\operatorname{Spec} k[x]$. It follows that the support of M is zero-dimensional.

We note that the additive group of k (considered as a discrete group) acts on $k[x]$ over k. By naturality, it acts on the $k[x]$-module M and therefore stabilizes the support of M. If k is an infinite field, this contradicts the fact that M has zero-dimensional support. Consequently, k is finite and therefore perfect. Let k' denote the residue field of $k[x]$ at some point of the support of M. The natural map $e' : k[x] \to k'$ is surjective, so that $\psi_{e'}$ is an equivalence. If we can show that $\psi_{e' \circ p}$ is an equivalence, then it will follow that $K_p \otimes_{k[x]} k' = 0$, a contradiction.

To show that $\psi_{e'^{op}}$ is an equivalence, we use the fact that $\mathcal{F}$ is an étale sheaf. Since $\operatorname{Spec} k' \to \operatorname{Spec} k$ is a surjection for the étale topology, it suffices to show that for any étale k'-algebra A, the map ψ_q is an equivalence where $q : A \to A \otimes_k k'$. But since k' is a Galois extension of k, q is simply given by the diagonal embedding of A into a product of copies of A, in which case the result is obvious.

We have now shown that there exists a cotangent complex $L_{\mathcal{F}/R}(\eta)$ at any point $\eta \in \mathcal{F}(C)$ such that C is truncated and almost of finite presentation over R, and that $L_{\mathcal{F}/R}$ is compatible with base change. Using the fact that $\mathcal{F}$ commutes with filtered colimits when restricted to m-truncated objects, we may construct $L_{\mathcal{F}/R}(C)$ whenever C is n-truncated, and using the fact that $\mathcal{F}(C) = \lim\{\mathcal{F}(\tau_{\leq m} C)\}$ we may extend the definition to all $C \in \mathcal{SCR}$. This completes the proof. □

7.5 Reformulations of the Representability Theorem

Combining Theorem 7.1.6 with Theorem 7.4.1, we deduce the following version of the representability criterion:

Theorem 7.5.1. *Let $\mathcal{F} : \mathcal{SCR} \to \mathcal{S}$ be a functor, and $p : \mathcal{F} \to \operatorname{Spec} R$ a natural transformation. Suppose that R is a Noetherian derived G-ring with a dualizing module. Then $\mathcal{F}$ is representable by a derived n-stack which is almost of finite presentation over R if and only if the following conditions are satisfied:*

1. *Finite Presentation: The functor $\mathcal{F}$ commutes with filtered colimits when restricted to $\tau_{\leq k}\mathcal{SCR}$, for each $k \geq 0$.*

2. *Truncatedness: The space $\mathcal{F}(A)$ is n-truncated for any discrete commutative ring A.*

3. *Descent: The functor $\mathcal{F}$ is a sheaf with respect to the étale topology.*

4. *Cohesiveness: If $A \to C$ and $B \to C$ are surjective maps in $\mathcal{SCR}$, then the natural map $\mathcal{F}(A \times_C B) \to \mathcal{F}(A) \times_{\mathcal{F}(C)} \mathcal{F}(B)$ is an equivalence.*

5. *Nilcompleteness: For any $A \in \mathcal{SCR}$, the natural map $\mathcal{F}(A) \to \lim\{\mathcal{F}(\tau_{\leq k} A)\}$ is an equivalence.*

6. *Representability of Formal Deformations: If A is a (discrete) commutative ring which is complete, local, and Noetherian, then the natural map $\mathcal{F}(A) \to \lim\{\mathcal{F}(A/\mathfrak{m}^k)\}$ is an equivalence, where $\mathfrak{m}$ denotes the maximal ideal of A.*

7. *Finite Dimensionality: Let $\eta \in \mathcal{F}(C)$, where C is a (discrete) integral domain which is finitely generated as a $\pi_0 R$-algebra. For each $i \in \mathbf{Z}$, the tangent module $T_i(\eta)$ is a finitely generated C-module.*

Remark 7.5.2. Although the statement of Theorem 7.5.1 may look complicated because of its many hypotheses, one should keep in mind that conditions (1) through (5) are automatically satisfied in almost any case of interest.

Remark 7.5.3. Theorem 7.5.1 may appear more concrete than Theorem 7.1.6 because of the absence of the cotangent complex, but it is harder to apply in practice. It is hard to imagine a way of computing the tangent modules $T_i(\eta)$ which does not involve the cotangent complex (which, after all, is simply a means of fitting the tangent modules $T_i(\eta)$ together into a base-change compatible package).

Remark 7.5.4. Suppose that, in condition (7), the algebra C is not finitely generated as an R-algebra but is instead the completion of some finitely generated R-algebra at a maximal ideal. Let $\mathfrak{m}$ denote the maximal ideal of C. Using conditions (4) through (6), we can deduce the finite generation of the tangent modules $\{T_i(\eta)\}_{i\in\mathbb{Z}}$ from the finite-dimensionality of the vector spaces $\{T_i(\eta_0)\}$, where $\eta_0 \in T_i(C/\mathfrak{m})$ is the corresponding $C/\mathfrak{m}$-valued point of $\mathcal{F}$. However, to pass from the finite generation of these *formal* tangent modules to the finite generation required by the Theorem, one needs to know that a finite set of generators for the tangent module at some point can be extended over some neighborhood of that point so that they generate the tangent module globally. This requires some kind of "openness of versality" condition of the type discussed in [2]. We shall refrain from giving an exact formulation, since in practice it is much more natural to verify (7) by computing the cotangent complex of $\mathcal{F}$.

Chapter 8

Examples and Applications

In this final section, we give some examples of derived moduli spaces that can be constructed using Theorem 7.1.6. We will confine our attention to three basic examples: moduli spaces of (semi)stable curves, Picard schemes, and Hilbert schemes. Other examples will be given in [23], after we have developed the theory of *geometric stacks*.

We will work with the étale topology throughout this section. All derived schemes and derived stacks will be considered with respect to this topology.

8.1 Stable Curves

One of the simplest examples of derived moduli spaces is the derived moduli space of semistable curves (of some fixed genus g). As it turns out, this space may be constructed without even using Theorem 7.1.6, because the classical moduli stack of semistable curves already represents the appropriate functor on *all* derived schemes. We now make this precise.

Definition 8.1.1. A morphism $p : X \to S$ of derived stacks is a *semistable curve of genus* g if p is a relative algebraic space which is bounded, flat, almost of finite presentation, and each geometric fiber $X \times_S \operatorname{Spec} k$ is a semistable curve of genus g over $\operatorname{Spec} k$ in the usual sense. That is, $X \times_S \operatorname{Spec} k$ is connected, one dimensional, of arithmetic genus g, and has at worst nodal singularities.

For each $A \in \mathcal{SCR}$, we let $\mathfrak{M}_g(A)$ denote the ∞-groupoid of semistable curves of genus g over $\operatorname{Spec} A$.

It is easy to see that $\mathfrak{M}_g(A)$ is essentially small for each $A \in \mathcal{SCR}$, so that we may regard $\mathfrak{M}_g$ as a functor $\mathcal{SCR} \to \mathcal{S}$.

In order to study the deformation theory of $\mathfrak{M}_g$, we need the following global version of Proposition 3.3.8:

Proposition 8.1.2. *Let $p : X \to \operatorname{Spec} A$ be a relative derived scheme, and let M be a connective A-module. Then the ∞-groupoid consisting of relative derived schemes $X' \to \operatorname{Spec} A \oplus M$ equipped with an equivalence $X \simeq X' \times_{\operatorname{Spec} A \oplus M} \operatorname{Spec} A$ is classified by the space*

$\mathrm{Hom}_{\mathrm{QC}_X}(L_{X/A}, p^*M[1])$. *Moreover, if* X *is flat over* $\operatorname{Spec} A$, *then any such* X' *is flat over* $\operatorname{Spec}(A \oplus M)$.

Proof. For the first part, we observe that the formation of the small extension classified by the space $\mathrm{Hom}_{\mathrm{QC}_X}(L_{X/A}, p^*M[1])$ gives a map which is an equivalence locally on X by Proposition 3.3.8. Since both sides are sheaves on X, the conclusion follows. For the second part, it suffices to work locally on X, so that we may suppose that $X' = \operatorname{Spec} B'$. It suffices to prove that if N is a discrete $A \oplus M$-module, then $B' \otimes_{A \oplus M} N$ is discrete. Filtering N if necessary, we may reduce to the case where $\pi_0 M$ acts trivially on N, so that N may be regarded as an A-module. Then $B' \otimes_{A\oplus M} N \simeq B' \otimes_{A \oplus M} A \otimes_A N$ is discrete because

$$B' \otimes_{A \oplus M} A$$

is assumed flat over A. □

Theorem 8.1.3. *The moduli functor* $\mathfrak{M}_g$ *is representable by a derived* 0*-stack which is smooth over* $\operatorname{Spec} \mathbf{Z}$.

Proof. The classical theory of moduli of curves tells us that there exists an ordinary Deligne-Mumford stack X, smooth over $\mathbf{Z}$, which represents $\mathfrak{M}_g$ on all ordinary commutative rings. We may regard X as a 0-truncated derived scheme. The tautological semistable curve over X gives rise to a transformation $X \to \mathfrak{M}_g$. It now suffices to show that the induced map $X(A) \to \mathfrak{M}_g(A)$ is an equivalence for all A. Since both functors are nilcomplete, we may reduce to the case where A is n-truncated. We may now work by induction on n. If $n = 0$, the claim follows from the definition of X. For $n > 0$, we may view A as a square-zero extension of $\tau_{\leq n-1} A$. It therefore suffices to show that X is formally étale over $\mathfrak{M}_g$. For this, we simply compute the cotangent complex of $\mathfrak{M}_g$ using Proposition 8.1.2 and observe that it agrees with the cotangent complex of X. □

Remark 8.1.4. We could also deduce Theorem 8.1.3 directly from Theorem 7.1.6. The only difficult points to check are conditions (3) and (4). The verification of (4) involves computing the cotangent complex of $\mathfrak{M}_g$, which is essentially equivalent to completing the deformation-theoretic calculations that are needed in the above proof. The verification of (3) uses the (classical or derived) Grothendieck existence theorem.

Remark 8.1.5. Given the basic moduli space $\mathfrak{M}_g$, one can construct all manners of variations: moduli spaces of smooth or stable curves (which are open subfunctors of $\mathfrak{M}_g$), moduli spaces of pointed curves, moduli spaces of stable maps, and so forth.

Our job in this section was particularly easy because the classical moduli stack of semistable curves was already smooth. This will not be the case in the other examples that we consider, and in these cases the underlying 0-truncated derived stack of the classical solution to the moduli problem will *not* represent the moduli functor in general. In these cases we will actually *need* to use Theorem 7.1.6 in order to construct the moduli stack.

8.2 Derived Picard Schemes

Let $p : X \to S$ be morphism of schemes. The classical Picard functor is sometimes defined to be the sheafification of the presheaf which assigns to each S-scheme S' the group of isomorphism classes of line bundles on $X' = X \times_S S'$. This sheafification procedure is rather ad-hoc, and there are various methods available for avoiding it. If p has a section s, one can instead define $\mathrm{Pic}_{X/S}(S')$ to be the category of line bundles $\mathcal{L}$ on X' equipped with a trivialization of $s^* \mathcal{L}$. Under suitable assumptions on the morphism p, this category will be discrete (in the sense that there are no nontrivial automorphisms of any object), and one can prove in this case that $\mathrm{Pic}_{X/S}$ is representable by an algebraic space (this is one of the original applications of Artin's representability theorem: see [2]). We will opt for a different approach, and consider instead the *Picard stack* $\widetilde{\mathrm{Pic}}_{X/S}$, which assigns to each $S' \to S$ the groupoid of line bundles over X'. This definition makes perfectly good sense in the case where S, S', and X are derived schemes. A *line bundle* on X is an object of QC_X which is locally free of rank 1. The line bundles on X form a small ∞-groupoid, which we shall denote by $\mathrm{Pic}(X)$. We then set $\widetilde{\mathrm{Pic}}_{X/S}(S') = \mathrm{Pic}(X')$.

In the case where $X = S$, the functor $\widetilde{\mathrm{Pic}}_{X/S}$ is representable by the classifying stack of the multiplicative group. In general, $\widetilde{\mathrm{Pic}}_{X/S}$ is given by the Weil restriction $p_* \widetilde{\mathrm{Pic}}_{X/X}$ of $\widetilde{\mathrm{Pic}}_{X/X}$. In [23], we will prove the representability of Weil restrictions such as this one in great generality. For the moment, we will be content to make the following observation:

Proposition 8.2.1. *Let $p : X \to S$ be a proper, flat, relative algebraic space. Let $\mathcal{F} \to X$ be a natural transformation of functors, and suppose that $L_{\mathcal{F}/X}$ exists and is almost perfect. Then the Weil restriction $p_* \mathcal{F}$ has an almost perfect cotangent complex over S.*

Proof. Let $\eta_0 \in S(B)$ and let M be a connective B-module. Then the fiber of $p_* \mathcal{F}(B \oplus M) \to (p_* \mathcal{F})(B) \times_{S(B)} S(B \oplus M)$ is also given by the fiber of $\mathcal{F}(X_{B\oplus M}) \to \mathcal{F}(X_B) \times_{X_B} X_{B\oplus M}$. By assumption, this is given by

$$\mathrm{Hom}_{\mathrm{QC}_{X_B}}(L_{\mathcal{F}/X}|X_B, p^* M|X_B).$$

Since the restriction of p to X_B is a proper, flat, relative algebraic space over $\mathrm{Spec}\, B$, we may deduce the existence of $L_{p_* \mathcal{F}/S}$ from Corollary 5.5.8. □

Theorem 8.2.2. *Suppose $p : X \to S$ is a proper, flat, relative algebraic space. Then $\widetilde{\mathrm{Pic}}_{X/S}$ is representable by a derived stack which is locally of finite presentation over S.*

Proof. Without loss of generality, we may suppose that $S = \mathrm{Spec}\, A$ is affine. Using Corollary 5.4.7 we may reduce to the case where A is n-truncated. Proposition 5.4.10 may be used to reduce further to the case where A is of finite presentation over $\mathbf{Z}$, and therefore a derived G-ring. We may now apply Theorem 7.1.6, once we have verified its hypotheses. The only conditions which offer any difficulty are (3) and (4). Condition (3) follows from the (classical or derived) formal GAGA theorem, while condition (4) follows from Proposition 8.2.1. □

Remark 8.2.3. The proof of Proposition 8.2.1 also shows how to compute the cotangent complex of a Weil restriction $p_* \mathcal{F}$: it is the quasi-coherent complex on $p_* \mathcal{F}$ given by applying Corollary 5.5.8 to $L_{\mathcal{F}/X}$. Morally, this is given by taking the dual of $L_{\mathcal{F}/X}$, pushing it forward to $p_* \mathcal{F}$, and taking the dual again. This idea can be made precise using a derived version of Grothendieck duality.

Let $p : X \to S$ be a proper flat relative algebraic space equipped with a section s. Then p and s induce pullback maps $p^* : \widetilde{\mathrm{Pic}}_{S/S} \to \widetilde{\mathrm{Pic}}_{X/S}$ and $s^* : \widetilde{\mathrm{Pic}}_{X/S} \to \widetilde{\mathrm{Pic}}_{S/S}$. One can then define a *reduced Picard functor* $\mathrm{Pic}_{X/S}$ to be the kernel of s^*. Using the pullback functor p^* and the group structure on $\widetilde{\mathrm{Pic}}_{X/S}$, we deduce a natural decomposition $\widetilde{\mathrm{Pic}}_{X/S} \simeq \mathrm{Pic}_{X/S} \times \widetilde{\mathrm{Pic}}_{S/S}$. Consequently, $\mathrm{Pic}_{X/S}$ may also be defined as the cokernel (in the étale topology) of p^*, and is independent of s. The cotangent complex of $\mathrm{Pic}_{X/S}$ is dual to $K[1]$, where K is the cokernel of the natural map $\mathcal{O}_S \to p_* \mathcal{O}_X$. In particular, if K has Tor-amplitude ≤ -1, then the cotangent complex of $\mathrm{Pic}_{X/S}$ is connective so that $\mathrm{Pic}_{X/S}$ is representable by a derived scheme. We note that the formation of K is compatible with arbitrary base change, so that the condition on its Tor-amplitude can be checked after base change to every geometric point $\operatorname{Spec} k \to S$; after this base change, it is equivalent to the assertion that $H^0(X, \mathcal{O}_X) \simeq k$.

Remark 8.2.4. Under the same conditions, the classical Picard functor is representable by an algebraic space. This algebraic space is obtained from our derived version $\mathrm{Pic}_{X/S}$ by truncating the structure sheaf. In particular, they have the same étale topologies, and so any topological question concerning $\mathrm{Pic}_{X/S}$ (such as whether or not it is separated, or whether its connected components are quasi-compact) is equivalent to the classical analogue of the same question.

Remark 8.2.5. The tensor product of line bundles induces an E_∞-multiplication (with inverses) on each of the Picard functors introduced above. We may therefore think of $\mathrm{Pic}_{X/S}$ as a "commutative group object" in the setting of derived stacks over S. Consequently, the cotangent complex Ω of $\mathrm{Pic}_{X/S}$ is the pullback of its restriction to S along the identity section (just as the tangent bundle to any Lie group has a canonical trivialization by left-invariant vector fields).

Remark 8.2.6. Let S be an ordinary scheme, and let G be a group scheme over S. If G is flat over S, then the derived fiber powers of G over S coincide with the ordinary fiber powers of G over S. It follows that G may be given the structure of a "group object" (with A_∞-multiplication) in the setting of derived S-schemes. If G is not flat over S, then this need not be true: the multiplication $\tau_{\leq 0}(G \times_S G) \to G$ need not extend to $G \times_S G$ in any canonical way. In this setting, the cotangent complex of G need not be a pullback from S. For example, one may have a group scheme which is smooth along the identity section, but not everywhere smooth. This phenomenon cannot arise in the derived setting.

Example 8.2.7. Let $p : X \to S$ be a proper flat morphism with geometrically connected, geometrically reduced fibers of dimension 1. By reducing to the case where S is the spectrum of a field, one can easy show that $L_{\mathrm{Pic}_{X/S}/S}$ is locally free (its fiber at a point $\mathrm{Spec}\, k \to L_{\mathrm{Pic}_{X/S}/S}$ is equivalent to the vector space $\mathrm{H}^1(X \times_S \mathrm{Spec}\, k, \mathcal{O}_X \,|X \times_S \mathrm{Spec}\, k))$. Consequently, $\mathrm{Pic}_{X/S}$ is smooth, and therefore flat over S. When S is an ordinary scheme, this means that $\mathrm{Pic}_{X/S}$ may be identified with the classical Picard scheme.

Example 8.2.8. Let $p : X \to S$ be an abelian variety (over an ordinary scheme S, say) of dimension $n > 1$. Then the cotangent complex of $\mathrm{Pic}_{X/S}$ is not projective (for example, the restriction of $\pi_1 L_{\mathrm{Pic}_{X/S}/S}$ along the identity section is a vector bundle of dimension $\frac{n^2-n}{2}$). Consequently $\mathrm{Pic}_{X/S}$ is not smooth over S, even along its identity section. It follows that we cannot identify the identity component of $\mathrm{Pic}_{X/S}$ with the dual abelian variety $X^\vee$ of X.

This is a case in which a classical moduli problem has multiple derived analogues. There is a natural map $j : X^\vee \to \mathrm{Pic}_{X/S}$ which identifies $X^\vee$ with the 0-truncation of the identity component of $\mathrm{Pic}_{X/S}$.

To understand why j is not étale, we must recall a few simple facts about line bundles on abelian varieties. Suppose that S is an ordinary scheme. Given a line bundle $\mathcal{L}$ of degree 0 on X which is trivialized along the identity section, there is a unique isomorphism $\phi : m^* \mathcal{L} \simeq \pi_0^* \mathcal{L} \otimes \pi_1^* \mathcal{L}$ which is compatible with various trivializations along the identity sections (here $m : X \times_S X \to X$ represents the addition law and $\pi_0, \pi_1 : X \times_S X \to X$ the two projections). This isomorphism gives rise to a group structure on the complement of the identity section in the total space of $\mathcal{L}$, and therefore an extension of X by the multiplicative group $\mathbf{G}_m$.

This argument fails if S is allowed to be a derived scheme, due to interactions between the higher homotopy groups of the structure sheaf of S and the higher cohomologies of abelian varieties. The derived scheme $\mathrm{Pic}_{X/S}$ classifies arbitrary line bundles on X (trivialized along the zero section), and its definition does not require any mention of the abelian variety structure on S. The dual abelian variety $X^\vee$ instead classifies extensions of X by $\mathbf{G}_m$ (as $\mathbf{Z}$-modules). For more details, we refer the reader to [25].

8.3 Derived Hilbert Scheme

Our last application of Theorem 7.1.6 will be the construction "derived Hilbert schemes". Recall that in classical algebraic geometry, given a separated morphism $X \to S$ of schemes, the Hilbert functor $\mathrm{Hilb}_{X/S}(S')$ is defined to be the *set* of closed subschemes $Y \subseteq X \times_S S'$ which are proper, flat and of finite presentation over S'. A classical result of Grothendieck asserts that $\mathrm{Hilb}_{X/S}$ is representable by a scheme if X is projective over S. Moreover, in this case $\mathrm{Hilb}_{X/S}$ may be decomposed into disjoint components (classified by their Hilbert polynomials), each of which is quasi-compact.

Using his abstract representability criteria, Artin was able to prove the representability of the Hilbert functor under much weaker assumptions. However, the price of using the abstract approach is that it gives no information about the global structure of the Hilbert functor: one knows only that $\mathrm{Hilb}_{X/S}$ is representable by an algebraic space. Quasi-compactness may fail, even for its connected components.

The moduli problem represented by the Hilbert functor should be thought of in two parts: first, one classifies all proper, flat S-schemes Y. Having done this, one considers all closed immersions from Y into X (over S). The first part of the problem is more basic, but the second part is relevant for two reasons:

1. The collection of proper flat S-schemes is naturally organized into a category. It is unwise to ignore the existence of nontrivial automorphisms in this category. Adding the data of an embedding into X kills all of these automorphisms, thus "rigidifying" the moduli problem.

2. By restricting our attention to S-schemes which arise as closed subschemes of X, we avoid certain technical issues concerning the algebraization of formal deformations. There can be no algebraic stack which classifies proper, flat families: the existence of such a stack would imply that every proper, flat formal scheme was algebraic. This issue does not arise for subschemes of a given ambient scheme which is already algebraic (because Grothendieck's formal GAGA theorem implies that when S is the spectrum of a complete Noetherian ring, then any *formal* closed subscheme of the formal completion of X is the formal completion of a closed subscheme of X).

In the derived context, (1) becomes somewhat irrelevant. The Hilbert functor will be $\mathcal{S}$-valued, rather than set-valued, whether we rigidify the moduli problem or not. However, (2) is just as much an issue in the derived context as in the classical context, and thus we shall continue to restrict our attention to the classification of closed subschemes of some fixed X. Let us simply remark that the condition that Y be embedded in X can be somewhat relaxed: we could equally well consider a Hilbert-like moduli functor which classified flat S'-derived schemes which were finite over X', for example.

Definition 8.3.1. Let $p : X \to S$ be a separated relative algebraic space. The *derived Hilbert functor* $\mathrm{Hilb}_{X/S}$ associates to each derived S-scheme S' the ∞-groupoid of derived $X' = X \times_S S'$-schemes Y which are proper and flat over S', and for which $Y \to X'$ is a closed immersion which is almost of finite presentation.

It not difficult to see that $\mathrm{Hilb}_{X/S}$ is $\mathcal{S}$-valued (that is, given S' as above, there are a bounded number of possibilities for Y up to equivalence), and that $\mathrm{Hilb}_{X/S}$ is a sheaf for the étale topology. We should warn the reader that, unlike the classical Hilbert functor, $\mathrm{Hilb}_{X/S}$ is not *set-valued.* This is because a closed immersion of derived schemes need not be a categorical monomorphism in any reasonable sense.

Remark 8.3.2. If S is a derived algebraic space, then on discrete commutative rings $\mathrm{Hilb}_{X/S}$ agrees with the classical Hilbert functor associated to the map of algebraic spaces $\tau_{\leq 0}X \to \tau_{\leq 0}S$. Thus, if $\mathrm{Hilb}_{X/S}$ is representable by a derived stack, then it is representable by a derived algebraic space.

Theorem 8.3.3. *Suppose that $p : X \to S$ is a separated relative algebraic space. Then* $\mathrm{Hilb}_{X/S}$ *is representable by a relative derived algebraic space which is locally almost of finite presentation over S.*

Proof. Without loss of generality, we may suppose that $S = \operatorname{Spec} A$ is affine. We note that if U is an open subfunctor of X, then $\mathrm{Hilb}_{U/S}$ is an open subfunctor of $\mathrm{Hilb}_{X/S}$. Moreover, if X is the filtered colimit of a system of open subfunctors U_α, then $\mathrm{Hilb}_{X/S}$ is the filtered colimit of the open subfunctors $\mathrm{Hilb}_{U_\alpha/S}$. Consequently, we may reduce to the case where X is quasi-compact over S, and therefore bounded.

We note that if S' is n-truncated, then the S'-valued points of $\mathrm{Hilb}_{X/S}$ depend only on $\tau_{\leq n}X$. Using a direct limit argument, we may find a map $A_0 \rightarrow A$, where A is of finite presentation over $\mathbf{Z}$, and a derived scheme X_0 almost of finite presentation over $S_0 = \operatorname{Spec} A_0$ together with an equivalence $\tau_{\leq n}X \simeq \tau_{\leq n}X_0 \times_{S_0} S$. Enlarging A_0 if necessary, we may guarantee that X_0 is separated.

Using Corollary 5.4.7, we may deduce the representability of $\mathrm{Hilb}_{X/S}$ from the representability of Hilb_{X_0/S_0} for all $n \geq 0$. Thus, we may suppose that A is of finite presentation over $\mathbf{Z}$, and therefore a derived G-ring.

We now apply Theorem 7.1.6 to the functor $\mathrm{Hilb}_{X/S}$. As usual, the only conditions which are not obvious are (3) and (4). Condition (3) follows from the classical Grothendieck existence theorem (we could also prove the derived version of condition (3) using our derived version of Grothendieck's formal GAGA theorem).

It remains to verify condition (4). We will do this by computing the cotangent complex of $\mathrm{Hilb}_{X/S}$ at a point $\eta \in \mathrm{Hilb}_{X/S}(B)$. The point η classifies a closed immersion $Y \rightarrow X' = X \times_S \operatorname{Spec} B$ such that the induced map $q : Y \rightarrow S'$ is flat. Proposition 8.1.2 implies that if M is a connective B-module, then the fiber of the natural map $\mathrm{Hilb}_{X/S}(B \oplus M) \rightarrow \mathrm{Hilb}_{X/S}(B)$ over η is given by the space

$$\mathrm{Hom}_{\mathrm{QC}_Y}(L_{Y/X'}, q^* M).$$

This functor of M is corepresentable by an almost perfect B-module by Proposition 5.5.8. □

Remark 8.3.4. In characteristic zero, a derived version of the Hilbert scheme (of projective space) has been constructed by Ciocan-Fontanine and Kapranov (see [7]) using a different approach.

Chapter 9

Appendix: Grothendieck Topologies on ∞-Categories

In this appendix, we sketch the construction of an ∞-topos from a small ∞-category with a Grothendieck topology. A related (and more detailed) discussion can be found in [39]. However, the theory presented here is slightly different because we will impose weaker descent conditions, and the ∞-topoi that we construct will not necessarily be t-complete in the sense of [39]. The ∞-topoi constructed in [39] may be obtained from ours by passing to the t-completion. On the other hand, the ∞-topoi constructed here may be described by a simple universal property (see Proposition 9.0.9 below).

Let $\mathcal{C}$ be a small ∞-category. We now recall the definition of a Grothendieck topology on $\mathcal{C}$.

If $X \in \mathcal{C}$ is an object, then a *sieve* on X is a full subcategory $S \subseteq \mathcal{C}_{/X}$, which is closed downwards in the sense that if a morphism $Y \to X$ belongs to S and $Z \to Y$ is any morphism, then the composite $Z \to X$ belongs to S.

If $f : X \to Y$ is a morphism and S is a sieve on Y, then we may define a sieve f^*S on X by declaring that a morphism $Z \to X$ belongs to f^*S if the composite map $Z \to Y$ belongs to S.

A *Grothendieck topology* on $\mathcal{C}$ consists of the specification, for each object $X \in \mathcal{C}$, of a distinguished family of sieves on X which are called *covering sieves*. The collection of covering sieves is required to satisfy the following conditions:

1. For any object $X \in \mathcal{C}$, the sieve consisting of all morphisms $\{Y \to X\}$ is covering.

2. If $f : X \to Y$ is a morphism and S is a covering sieve for Y, then f^*S is a covering sieve for S.

3. Suppose that S is a sieve on $X \in \mathcal{C}$, and S' is a covering sieve for X. Suppose further that for each $f : Y \to X$ belonging to S', the sieve f^*S on Y is covering. Then S is covering.

Remark 9.0.5. There is a natural bijection between the set of all sieves on an object $X \in \mathcal{C}$ and the set of all sieves on the corresponding object in the homotopy category $h\,\mathcal{C}$. Consequently, we see that specifying a Grothendieck topology on $\mathcal{C}$ is equivalent to specifying a Grothendieck topology on $h\,\mathcal{C}$.

Now suppose that the ∞-category $\mathcal{C}$ has been equipped with a Grothendieck topology. A presheaf $\mathcal{F} : \mathcal{C}^{op} \to \mathcal{S}$ will be called a *sheaf* if it satisfies the following condition: for any object $X \in \mathcal{C}$, and any covering sieve S on X, the natural map

$$\mathcal{F}(X) \to \lim_{Y \in S} \mathcal{F}(Y)$$

is an equivalence.

Proposition 9.0.6. *Let $\mathcal{C}$ be a small ∞-category equipped with a Grothendieck topology. The ∞-category $\mathrm{Shv}(\mathcal{C})$ of sheaves on $\mathcal{C}$ forms an ∞-topos.*

Proof. It will suffice to show that $\mathrm{Shv}(\mathcal{C})$ is a left-exact localization of the ∞-category $\mathcal{P} : \mathcal{S}^{\mathcal{C}^{op}}$ of presheaves on $\mathcal{C}$. We sketch the construction of the localization functor $L : \mathcal{P} \to \mathrm{Shv}(\mathcal{C}) \subseteq \mathcal{P}$; it parallels the construction given in [22] in the case where $\mathcal{C}$ is an ordinary category.

If $\mathcal{F}$ is a presheaf on $\mathcal{C}$, we let $\mathcal{F}^+(X) = \mathrm{colim}_S \lim_{Y \in S} \mathcal{F}(Y)$. Here, the colimit is taken over the filtered collection of all sieves S on X, and the limit is taken over the sieve S (regarded as an ∞-category). The construction $\mathcal{F} \mapsto \mathcal{F}^+$ may be regarded as a functor $\mathcal{P} \to \mathcal{P}$. There is a transformation $\mathcal{F} \to \mathcal{F}^+$ (natural in $\mathcal{F}$).

We will construct L as the colimit of a transfinite sequence of iterations of the functor $\mathcal{F} \to \mathcal{F}^+$. Namely, we define functors $L_\alpha : \mathcal{P} \to \mathcal{P}$ indexed by the ordinals α by transfinite recursion. Let L_0 be the identity functor, let $L_{\alpha+1}\,\mathcal{F} = (L_\alpha\,\mathcal{F})^+$, and let $L_\lambda\,\mathcal{F} = \mathrm{colim}_{\beta<\lambda} L_\beta\,\mathcal{F}$ when λ is a limit ordinal.

One proves by induction on α that $\mathrm{Hom}_{\mathcal{P}}(L_\alpha\,\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}_{\mathcal{P}}(\mathcal{F}, \mathcal{G})$ whenever $\mathcal{G}$ is a sheaf. Using straightforward cardinality estimates, one shows that $L_\alpha\,\mathcal{F}$ is a sheaf for α sufficiently large (independent of $\mathcal{F}$). We may then take $L = L_\alpha$. One checks easily that L is an accessible functor. To prove that L is left-exact, it suffices to show that the functor L^+ is left-exact. This follows from the fact that the partially ordered set of sieves on any given object $X \in \mathcal{C}$ is directed downward under inclusion (in fact, it is closed under finite intersections: this follows easily from the definition). □

Remark 9.0.7. The underlying topos of discrete objects $\tau_{\leq 0}\,\mathrm{Shv}(\mathcal{C})$ is naturally equivalent to the category of sheaves of sets on the homotopy category $h\,\mathcal{C}$.

Remark 9.0.8. In contrast to the classical theory of Grothendieck topologies, it is not the case that every ∞-topos arises as the ∞-category of sheaves on some small ∞-category with a Grothendieck topology.

The ∞-topos $\mathrm{Shv}(\mathcal{C})$ admits the following characterization:

Proposition 9.0.9. *Let $\mathcal{C}$ be a small ∞-category equipped with a Grothendieck topology, and let $\mathcal{X}$ be any ∞-topos. Then the ∞-category of geometric morphisms from $\mathcal{X}$ to $\mathrm{Shv}(\mathcal{C})$ is equivalent to the full subcategory of geometric morphisms $f : \mathcal{X} \to \mathcal{S}^{\mathcal{C}^{op}}$ which possess the following property:*

- *For each object $X \in \mathcal{C}$ and each covering sieve S of X, the induced map*

$$\coprod_{X' \in S} f^* X' \to f^* X$$

is a surjection in $\mathcal{X}$.

Here we have identified objects of $\mathcal{C}$ with the corresponding representable presheaves on $\mathcal{C}$ via the Yoneda embedding.

Proof. Since $\mathrm{Shv}(\mathcal{C})$ is a localization of $\mathcal{S}^{\mathcal{C}^{op}}$, it is clear that the ∞-category of geometric morphisms from $\mathcal{X}$ into $\mathrm{Shv}(\mathcal{C})$ is a full subcategory of the ∞-category of geometric morphisms $\mathcal{X} \to \mathcal{S}^{\mathcal{C}^{op}}$. Moreover, $f : \mathcal{X} \to \mathcal{S}^{\mathcal{C}^{op}}$ belongs to this full subcategory if and only if the natural transformation $f^* \to f^* \circ L$ is an equivalence, where $L : \mathcal{S}^{\mathcal{C}^{op}} \to \mathcal{S}^{\mathcal{C}^{op}}$ denotes the corresponding localization functor.

Suppose first that f factors through $\mathrm{Shv}(\mathcal{C})$. Let S be a covering sieve of $X \in \mathcal{C}$. To prove that

$$\coprod_{X' \in S} f^* X' \to X$$

is surjective, it suffices to prove the corresponding statement in $\mathrm{Shv}(\mathcal{C})$, so we may assume that $\mathcal{X} = \mathrm{Shv}(\mathcal{C})$ and $f^* = L$. Let Y denote the image of $\coprod_\alpha LX_\alpha$ in X. Then $Y \to LX$ is (-1)-truncated. To prove that $Y \simeq LX$, it suffices to show that the tautological point $\eta \in LX(X)$ lifts (automatically uniquely) to a point of $Y(X)$. This assertion is local on X, so it suffices to prove that $\eta | X_\alpha \in LX(X_\alpha)$ lifts to a point of $Y(X_\alpha)$, which follows immediately from the factorization $LX_\alpha \to Y \to LX$.

For the converse, let us suppose that f satisfies the condition stated in the theorem. We must show that the natural map $f^*X \simeq f^*LX$ is an equivalence. It suffices to show that $\mathrm{Hom}_{\mathcal{X}}(f^*LX, Y) \simeq \mathrm{Hom}_{\mathcal{X}}(f^*X, Y)$ is an equivalence for each $Y \in \mathcal{X}$. In other words, we must show that $\mathrm{Hom}(LX, f_*Y) \to \mathrm{Hom}(LX, f_*Y)$. By definition, it will suffice to show that f_*Y is a sheaf. Equivalently, we must show that for each object $Z \in \mathcal{C}$ and each covering sieve S over Z, the natural map $\mathrm{colim}_{Z' \in S} f^*Z' \to f^*Z$ is an equivalence. We note that the colimit $\mathrm{colim}_{Z' \in S} Z'$ in $\mathcal{S}^{\mathcal{C}^{op}}$ is given by the presheaf $\mathcal{F}$, where $\mathcal{F}$ is the "characteristic function" of the subcategory $S \subseteq \mathcal{C}_{/Z}$. In other words, for each $C \in \mathcal{C}$, the space $\mathcal{F}(C)$ is given by the union of those components of $\mathrm{Hom}_{\mathcal{C}}(C, Z)$ consisting of morphisms $C \to Z$ which belong to S.

In particular, we note that the natural map $\mathcal{F} \to Z$ is (-1)-truncated, so that $f^*\mathcal{F} \to f^*Z$ is (-1)-truncated. To prove that it is an equivalence, it suffices to show that it is surjective. This follows immediately from the assumption since $\coprod_{Z' \in S} f^*Z' \to f^*Z$ factors through $f^*\mathcal{F}$. □

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